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On distributivity equations of implications and contrapositive symmetry equations of implications

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Abstract

To avoid combinatorial rule explosion in fuzzy reasoning, we recently obtained new solutions of the distributivity equation of implication $I(x, T_1(y, z)) = T_2(I(x, y), I(x, z))$. Here we study and characterize all solutions of the functional equations consisting of $I(x, T_1(y, z)) = T_2(I(x, y), I(x, z))$ and $I(x, y) = I(N(y), N(x))$ when T_1 is a continuous but non-Archimedean triangular norm, T_2 is a continuous and Archimedean triangular norm, I is an unknown function, and N is a strong negation. It should be noted that these results differ from the ones obtained by Qin and Yang when both T_1 and T_2 are continuous and Archimedean. Our methods are suitable for three other distributivity equations of implications closely related to those mentioned above.

Keywords: Fuzzy connectives; Fuzzy implications; Continuous Archimedean t-norms; Continuous t-norms; Distributivity equations of implications; Contrapositive symmetry equations of implications

1. Introduction

We recently obtained solutions to the following *distributivity equation of implication* [15]:

$$I\big(x, T_1(y, z)\big) = T_2\big(I(x, y), I(x, z)\big), \quad x, y, z \in [0, 1] \tag{1}$$

when T_1 is a continuous but non-Archimedean triangular norm, T_2 is a continuous and Archimedean triangular norm, and I is an unknown function. However, from a fuzzy logical point of view [3,4,6], Eq. (1) goes back to the equation

$$I\big(x, T(y, z)\big) = T\big(I(x, y), I(x, z)\big), \quad x, y, z \in [0, 1], \tag{2}$$

which was posed and discussed by Türksen et al. [20]. In fact, Eq. (2) may reduce the complexity of fuzzy "IF-THEN" rules of fuzzy systems and improve their applications [3,6,7,13,14].

To construct a new family of t-norms, Fodor investigated the equation [8]

$$I(x, y) = I\big(N(y), N(x)\big), \quad x, y \in [0, 1], \tag{3}$$

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which is called a *contrapositive symmetry equation of implication* and is important in fuzzy preference modeling and expert systems with incomplete information [2,17]. Later, Baczyński generalized some of the results of Türksen et al. into strict t-norms and studied the functional equations composed of (2) and (3) [1,2]. Then Yang and Qin fully characterized the previous functional equations when T is a strict t-norm [21].

Using a similar line of thinking, here we characterize all solutions of the functional equations consisting of (1) and (3) when T_1 is a continuous but non-Archimedean triangular norm, T_2 is a continuous and Archimedean triangular norm, I is an unknown function, and N is a strong negation. It should be noted that these results differ from those obtained by Qin and Yang when both T_1 and T_2 are continuous and Archimedean [21]. Our method can also be applied to three other distributivity equations of implications closely related to those mentioned above. Moreover, this paper will borrow from the results and the proof methods in [15] as indicated further.

The remainder of the paper is organized as follows. In Section 2 we present some results concerning basic fuzzy logic connectives used later in the paper. In Section 3 we investigate the functional equations consisting of (1) and (3) when T_1 is a continuous but non-Archimedean triangular norm and T_2 is a strict triangular norm. In Section 4, we repeat the investigation for the assumption that T_2 is a nilpotent triangular norm. Section 5 concludes.

2. Preliminaries

In this section, we recall basic notations and facts used later in the paper.

Definition 2.1. (See [11].) A binary function $T : [0, 1]^2 \to [0, 1]$ is called a *triangular norm* (t-norm for short) if it fulfills the following conditions for every $x, y, z \in [0, 1]$:

(i) $T(x, y) = T(y, x)$ (commutativity),
(ii) $T(T(x, y), z) = T(x, T(y, z))$ (associativity),
(iii) $T(x, y) \leqslant T(x, z)$ when $y \leqslant z$ (monotonicity), and
(iv) $T(x, 1) = x$ (boundary condition).

Definition 2.2. (See [11].) A t-norm T is said to be

(i) *Archimedean* if, for every $x, y \in (0, 1)$, there exists some $n \in \mathbf{N}$ such that $x_T^n < y$, where $x_T^n = T(\underbrace{x, x, \ldots, x}_{n \text{ times}})$;
(ii) *Strict* if it is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$, when $x \in (0, 1]$ and $y < z$; and
(iii) *Nilpotent* if it is continuous and if for each $x \in (0, 1)$ there exists some $n \in \mathbf{N}$ such that $x_T^n = 0$.

Remark 2.3.

(i) A continuous t-norm T is Archimedean if and only if $T(x, x) < x$ holds for all $x \in (0, 1)$ [10, Proposition 5.1.2].
(ii) If T is strict or nilpotent, then it must be Archimedean. The converse is also true when it is continuous [11, Theorem 2.18].

Theorem 2.4. *(See [12].) For a function $T : [0, 1]^2 \to [0, 1]$, the following statements are equivalent.*

(i) *T is a continuous Archimedean t-norm.*
(ii) *T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t : [0, 1] \to [0, \infty]$ with $t(1) = 0$ that is uniquely determined up to a positive multiplicative constant such that*

$$T(x, y) = t^{-1}\bigl(\min\bigl(t(x) + t(y), t(0)\bigr)\bigr), \quad x, y \in [0, 1]. \tag{4}$$

Remark 2.5. (See [11].)

(i) A t-norm T is strict if and only if each continuous additive generator t of T satisfies $t(0) = \infty$.
(ii) A t-norm T is nilpotent if and only if each continuous additive generator t of T satisfies $t(0) < \infty$.