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Short communication

A note on the definition of a generalized fuzzy normed space

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Abstract

We reconsider the definition of a generalized fuzzy normed space from [I. Goleţ, On generalized fuzzy normed spaces and coincidence point theorems, Fuzzy Sets and Systems 161 (2010) 1138–1144], showing that the mapping used in the homogeneity axiom must satisfy the multiplicative Cauchy functional equation. Therefore, generalized fuzzy normed spaces are, in fact, fuzzy p-normed spaces with $p > 0$.

Keywords: Generalized fuzzy normed space; t-Norm of H-type; p-Normed space

1. Introduction

The concepts of φ-normed space and generalized fuzzy normed space (or fuzzy φ-normed space) were introduced as generalizations of those of normed space and fuzzy normed space, by considering an appropriate function φ in the homogeneity axiom.

We recall these definitions, as given in [2]. Here $\varphi : \mathbb{R} \to \mathbb{R}$ is a function with the properties:

(i) $\varphi(-t) = \varphi(t)$ for every $t \in \mathbb{R}$;
(ii) $\varphi(1) = 1$;
(iii) φ is strictly increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 1. (See [2, Definition 2.3].) A *φ-normed space* is a pair $(L, \|\cdot\|)$, where L is a real vector space and $\|\cdot\|$ is a real valued mapping defined on L satisfying

(N1) $\|x\| \geqslant 0$ for all $x \in L$, and $\|x\| = 0$ if and only if $x = \theta$;
(N2) $\|\alpha x\| = \varphi(\alpha)\|x\|$ for all $x \in L$, $\alpha \in \mathbb{R}$, where φ is a function with the properties (i)–(iii);
(N3) $\|x + y\| \leqslant \|x\| + \|y\|$, for all $x, y \in L$.

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Definition 2. (See [2, Definition 2.1].) A *fuzzy φ-normed space* is a triple $(L, N, *)$, where L is a real vector space, $*$ is a continuous t-norm, and N is a mapping from $L \times [0, \infty)$ into $[0, 1]$ such that the following conditions hold:

(FN1) $N(x, 0) = 0$ for all $x \in L$;
(FN2) $N(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$ (the null vector);
(FN3) $N(\alpha x, t) = N(x, \frac{t}{\varphi(\alpha)})$ for all x in L, $\alpha \neq 0$ and $t > 0$;
(FN4) $N(x + y, t + s) \geqslant N(x, t) * N(y, s)$ for all $x, y \in L$ and $t, s \geqslant 0$;
(FN5) the mapping $N(x, .) : [0, \infty) \to [0, 1]$ is left continuous for all $x \in L$.

If $\varphi(t) = |t|$ one obtains the definition of a fuzzy normed space [1]. If $\varphi(t) = |t|^p$ with $p \in (0, 1]$, then the space is called a *fuzzy p-normed space* [2]. Note that fuzzy p-normed spaces can be regarded as an extension of (deterministic) p-normed spaces. For convenience, the definition of these spaces is recalled below (cf. [6]).

Definition 3. Let $p \in (0, 1]$. A p-normed space is a real vector space L endowed with a p-norm $\|\cdot\|_p$, that is, the following conditions are satisfied:

(1) $\|x\|_p \geqslant 0$ for all $x \in L$, and $\|x\|_p = 0$ if and only if $x = \theta$;
(2) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$ for all $x \in L$, $\alpha \in \mathbb{R}$;
(3) $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$, for all $x, y \in L$.

It is worth mentioning that every p-normed space $(L, \|\cdot\|_p)$ induces in a natural way a fuzzy p-normed space (L, N, Min) with

$$N(x, t) = \begin{cases} 0, & t \leqslant \|x\|, \\ 1, & t > \|x\|. \end{cases}$$

The aim of this paper is to show that the only generalized fuzzy normed spaces in the sense of Definition 2 are fuzzy p-normed spaces, with $p \in (0, \infty)$. Namely, we prove that the mapping φ from the above definitions must satisfy the multiplicative Cauchy functional equation $\varphi(uv) = \varphi(u)\varphi(v)$, and therefore it can only be of the form $\varphi(t) = |t|^p$, for some positive p. Although for a large class of spaces, including those endowed with the triangular norm Min, p must be in $(0, 1]$, we can also give an example of a fuzzy p-normed space with $p > 1$.

2. Main results

We will show that, if the function φ from the definition of a fuzzy φ-normed space has the property that $\varphi(ab) \neq \varphi(b)\varphi(a)$, for some $a, b \neq 0$, then $N(x, \cdot)$ is a constant mapping on $(0, \infty)$ for every $x \in L$. As such,

$$N\left(x, \frac{t}{\varphi(\alpha)}\right) = N\left(x, \frac{t}{|\alpha|}\right) \quad (x \in L, \alpha \neq 0, t > 0),$$

hence the axiom (FN3) is not different from that in the definition of a fuzzy normed space [1]. Therefore, in order to have a proper generalization of the homogeneity axiom, the equality $\varphi(ab) = \varphi(b)\varphi(a)$ must hold for every $a, b \neq 0$. As this equality obviously holds if either of a or b is 0, we conclude that φ must satisfy $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \mathbb{R}$. But it is well known (see e.g., [5, Theorem 1.49]), that the only nonconstant, even, continuous solution of the multiplicative Cauchy functional equation $\varphi(uv) = \varphi(u)\varphi(v)$ is given by $\varphi(t) = |t|^p$, for some positive p.

To prove our claim, let $(L, N, *)$ be a fuzzy φ-normed space and $x \in L$, $a, b > 0$ be given. From (FN3) it follows that

$$N(abx, t) = N\left(x, \frac{t}{\varphi(ab)}\right),$$

for all $t > 0$. On the other hand,

$$N(abx, t) = N\left(bx, \frac{t}{\varphi(a)}\right) = N\left(x, \frac{t}{\varphi(b)\varphi(a)}\right),$$

for all $t > 0$.