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On linear and quadratic constructions of aggregation functions

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Abstract

In the paper we introduce and discuss linear and quadratic constructions of aggregation functions based on an a priori given aggregation function. We focus our attention on linear and quadratic constructions transforming any aggregation function from the considered class of aggregation functions into a new aggregation function belonging to the same class. We study quadratic constructions of distinguished classes of aggregation functions with neutral element $\epsilon \in \{0, 1\}$, in particular, semi-copulas, quasi-copulas, copulas, triangular norms and their duals. In all cases, the final results fully characterize quadratic functions which are universal for quadratic constructions of aggregation functions of the considered type.

Keywords: Aggregation function; Linear construction; Quadratic construction; Quasi-copula; Semi-copula

1. Introduction

Aggregation functions are applied in almost each scientific branch dealing with numerical information. Aggregation means the fusion of several input values into one value which represents the input data. Aggregation processes appear in many different contexts and therefore, as is emphasized in many sources (e.g., in [5,12,22]), there is a need of having a variety of aggregation functions. A lot of construction methods of new aggregation functions have been proposed in the literature. Most of them are based on algebraic approaches applied to arguments and output values of the given aggregation functions. For a survey of construction methods of aggregation functions see, e.g., the monographs [5,12] or the chapters [6,19]. Some other construction methods are proposed in the papers [2,3,8,9,14,17,18], among others. Quite rarely the methods merging only the input values and the corresponding output of the a priori given aggregation function are applied. The aim of this paper is to propose and develop just such method for construction of aggregation functions of different types in dependence on the given original aggregation function. Note, that

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some primary results in this streaming can be found in [17,18], and the basic ideas of our approach were presented at AGOP'2013 in Pamplona.

In this paper, we restrict ourselves to *binary aggregation functions* on the unit interval, i.e., functions $A: [0,1]^2 \to [0,1]$ which are increasing in each variable and satisfy the boundary conditions $A(0,0)=0$ and $A(1,1)=1$. Note, that the increasing monotonicity of A need not be strict. The class of all binary aggregation functions will be denoted by $\mathcal{A}$. In the sequel the word "binary" will be omitted.

If for each input $(x,y)\in[0,1]^2$ we put $z=A(x,y)$, then the triplet (x,y,z) is in $[0,1]^3$. Considering a ternary function $F: \mathbb{R}^3 \to \mathbb{R}$, we can define the composite function $A_F: [0,1]^2 \to \mathbb{R}$ by

$$A_F(x,y) = F(x,y,z), \quad \text{where } z = A(x,y). \tag{1}$$

It is interesting to know under what conditions imposed on A and F, the obtained function A_F is an aggregation function, or more, an aggregation function which belongs to the same class as A does. In this paper we will deal with special kinds of F's only, namely with linear and quadratic functions. Constructions based on such functions will be called *linear and quadratic constructions of aggregation functions*, respectively. We focus on such linear and quadratic constructions whose results are always aggregation functions within the considered class of aggregation functions. Our ideas will be illustrated on several distinguished classes of aggregation functions having many applications.

The paper is organized as follows: In Section 2, we introduce the classes of aggregation functions we will be working with. The third section provides a short survey of linear constructions of aggregation functions. Section 4 is devoted to the quadratic constructions of aggregation functions: after a short general discussion, we will investigate in detail quadratic constructions of semi-copulas and quasi-copulas, and then also quadratic constructions of their duals. In all cases, the final results characterize all quadratic functions which are universal for quadratic constructions of aggregation functions of the considered type. Finally, several concluding remarks are added.

2. Classes of aggregation functions

We start with recalling several notions which are basic for our next study. We restrict ourselves to the basic definitions of special types of aggregation functions we will be working with, their other properties and relationships can be found, e.g., in the monographs [5,12] or in the chapters of monographs [6,19].

We say that an element $\epsilon\in[0,1]$ is a *neutral element* of an aggregation function A if for each $x\in[0,1]$ it holds $A(x,\epsilon)=A(\epsilon,x)=x$. Similarly, $\alpha\in[0,1]$ is an *annihilator* of A if for each $x\in[0,1]$, $A(x,\alpha)=A(\alpha,x)=\alpha$. The class of all aggregation functions with neutral element ϵ (annihilator α) will be denoted by $\mathcal{A}_{[\epsilon]}$ ($\mathcal{A}_{(\alpha)}$).

In this paper, a special attention will be devoted to *semi-copulas*, i.e., aggregation functions $S: [0,1]^2 \to [0,1]$ with neutral element $\epsilon = 1$. The class of all semi-copulas will be denoted by $\mathcal{A}_{[1]}$. More information on semi-copulas can be found, e.g., in [4,7]. Note that semi-copulas are also called conjunctors.

An aggregation function A is said to be *1-Lipschitz* if for all $x_1,x_2,y_1,y_2\in[0,1]$,

$$\left|A(x_1,y_1)-A(x_2,y_2)\right| \le |x_1-x_2|+|y_1-y_2|. \tag{2}$$

The class of all 1-Lipschitz aggregation functions will be denoted by $\mathcal{A}_L$. It is known [16] that for each $A\in\mathcal{A}_L$ and for all $(x,y)\in[0,1]^2$ it holds

$$\max\{0,x+y-1\} \le A(x,y) \le \min\{1,x+y\}. \tag{3}$$

An aggregation function $Q: [0,1]^2 \to [0,1]$ with neutral element $\epsilon=1$ and satisfying the 1-Lipschitz property (2) is called a *quasi-copula* [11]. The original definition of quasi-copulas was given in [1]. The set of all quasi-copulas will be denoted by $\mathcal{Q}$. Clearly, each quasi-copula is a semi-copula (but not vice-versa). Distinguished examples of quasi-copulas are aggregation functions W, Π and M defined by

$$W(x,y)=\max\{0,x+y-1\}, \qquad \Pi(x,y)=xy, \qquad M(x,y)=\min\{x,y\}.$$

In the framework of quasi-copulas, for each $Q\in\mathcal{Q}$, it holds $W\le Q\le M$. Clearly, quasi-copulas as being 1-Lipschitz aggregation functions, also satisfy the boundaries given in (3).

Important aggregation functions are copulas, originally introduced in [21]. A *copula* is an aggregation function $C: [0,1]^2 \to [0,1]$ with neutral element $\epsilon=1$, satisfying the 2-increasing property, i.e., for all $x_1,x_2,y_1,y_2\in[0,1]$ with $x_1\le x_2$ and $y_1\le y_2$,