Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 268 (2015) 93–109

FUZZY sets and systems

www.elsevier.com/locate/fss

Generalized extended fuzzy implications

Chun Yong Wang [a], Bao Qing Hu [b,*]

[a] *School of Mathematical Sciences, Shandong Normal University, Jinan 250014, PR China*
[b] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China*

Received 26 April 2013; received in revised form 12 February 2014; accepted 3 May 2014

Available online 22 May 2014

Abstract

This paper is devoted to investigate generalized extended fuzzy implications in accordance with the generalized extension principle. The algebraic properties of generalized extended fuzzy implications are discussed on different algebras of fuzzy truth values. On this basis, fuzzy-valued fuzzy implications are constructed from arbitrary fuzzy implications and t-norms. Moreover, a fuzzy-valued fuzzy implication is identical with a generalized extended fuzzy implication on the algebra of fuzzy values, when they are induced by exactly the same continuous fuzzy implication and same upper semicontinuous t-norm.

Keywords: Type-2 fuzzy sets; Type-2 fuzzy implications; Generalized extended fuzzy implications; Fuzzy values; Fuzzy-valued fuzzy implications

1. Introduction

Fuzzy implications are the main operations in fuzzy logic [17], generalizing the classical implications to fuzzy logic. Different fuzzy implications have constructed various structures of fuzzy logic systems, such as the basic logic [17] which was shown to be the logic of continuous t-norms and their R-implications [7], the monoidal t-norms based logic [11] which was shown to be the logic of lower semicontinuous t-norms and their R-implications and so on. Moreover, they had been extended to type-2 fuzzy sets [49] in [15] and type-2 fuzzy logic systems were proposed in [25] from the view of type reductions and centroids [27]. However, it had been pointed out in [15] that there do not exist adjoint pairs between extended t-norms and extended fuzzy implications.

Type-2 fuzzy sets were introduced by Zadeh in [49]. Type-2 fuzzy sets are special fuzzy sets equipped with fuzzy truth values mapping from the unit interval to itself. As a consequence, fuzzy sets and interval-valued fuzzy sets [43,49] are special cases of type-2 fuzzy sets, which were equivalently expressed [1,26,27,33,37,39] and had already been used in many aspects [8,24,33–38,42,44,45].

Operations on type-2 fuzzy sets were investigated in many literatures [3,4,9,10,12–15,18–24,26,28,29,32,39,41,46–48], such as extended t-(co)norms and extended fuzzy implications in accordance with Zadeh's extension principle.

* Corresponding author.
E-mail addresses: chunyong_wang@163.com (C.Y. Wang), bqhu@whu.edu.cn (B.Q. Hu).

http://dx.doi.org/10.1016/j.fss.2014.05.010
0165-0114/

In [15], Gera and Dombi proposed type-2 fuzzy implications and investigated extended operations, where fuzzy implication was limited to be S-implication (resp. R-implication) based on maximum (resp. minimum). The generalized extended operations were studied in [22,23,29,41] in accordance with generalized extension principle [49], where t-norms were supposed to be continuous. However, continuous t-norms are too special. Considering fuzzy implications do not satisfy associativity, we shall discuss here lower and upper semicontinuous t-norms. In this paper, we investigate generalized extended fuzzy implications on different algebras of fuzzy truth values, where t-norms are not necessarily continuous. However, when generalized extended fuzzy implications are closed on the algebra of convex fuzzy truth values, fuzzy implications need to be continuous [15]. Thus continuity of fuzzy implications is required, while generalized extended fuzzy implications are type-2 fuzzy implications on the algebra of convex normal fuzzy truth values. Since a fuzzy value can be equivalently represented by a nest of closed intervals, we shall express generalized extended fuzzy implications in the form of nests of closed intervals on the algebra of fuzzy values. On this basis, fuzzy-valued fuzzy implications induced by arbitrary fuzzy implications and t-norms are proposed on the algebra of fuzzy values, regardless of the continuity of fuzzy implications, and the lower (resp. upper) semicontinuity of t-norms. Furthermore, a fuzzy-valued fuzzy implication is identical with a generalized extended fuzzy implication on the algebra of fuzzy values, when they are induced by exactly the same continuous fuzzy implication and same upper semicontinuous t-norm.

The contents of the paper are organized as follows. In Section 2, we recall some fundamental concepts and related properties. In Section 3, we investigate generalized extended fuzzy implications on different algebras of fuzzy truth values. Especially, we study the sufficient and necessary conditions when a generalized extended fuzzy implication is a type-2 fuzzy implication. Section 4 discusses generalized extended fuzzy implications on the algebra of fuzzy values. Moreover, we construct fuzzy-valued fuzzy implications from arbitrary fuzzy implications and t-norms, regardless of the continuity of fuzzy implications, and the lower (resp. upper) semicontinuity of t-norms. In the final section, our research is concluded.

2. Preliminaries

Let X and Y be nonempty universes, $\mathscr{M}(X,Y)$ be the set of all mappings from X to Y and I denote the unit interval $[0,1]$. $\mathscr{M}(X,Y)$ is denoted as $\mathscr{M}(X)$, if $Y=I$. A binary operation $\top : I\times I\to I$ (resp. $\bot : I\times I\to I$) is called a *t-norm* (resp. *t-conorm*) on I if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0).

An involutive negator $N : I\to I$ is a decreasing function and satisfies $N(N(x))=x$ for all $x\in I$. Assume that $*$ and $\star$ are two binary operations on I, then they are said to be *dual* with respect to (w.r.t., for short) N, if for all $x,y\in I$, $N(x*y)=N(x)\star N(y)$.

A fuzzy set A is a mapping from X to I, i.e., $A\in\mathscr{M}(X)$. A is called a *fuzzy truth value*, if $A\in\mathscr{M}(I)$. The two sets $\emptyset$ and X are special elements in $\mathscr{M}(X)$, with $\emptyset(x)=0$ and $X(x)=1$ for all $x\in X$, respectively. The order relation on fuzzy sets is defined as $A\subseteq B \Leftrightarrow A(x)\leqslant B(x)$ for all $x\in X$. For all $a,b\in I$, a special fuzzy truth value b/a is defined as

$$(b/a)(x)=\begin{cases} b, & x=a,\\ 0, & x\neq a.\end{cases}$$

If $b=1$, then b/a is also denoted as $\bar{a}$ for short. An α-cut set of fuzzy set A on X is $A_\alpha=\{x\in X \mid A(x)\geqslant\alpha\}$ for all $\alpha\in I$. If $\alpha=1$, then A_α is called the kernel of A and denoted as $\mathrm{Ker}(A)$. A strong α-cut set of fuzzy set A is $A_{\bar{\alpha}}=\{x\in X \mid A(x)>\alpha\}$ for all $\alpha\in I$. The least upper bound of A is denoted by A^{sup}, i.e., $A^{\mathrm{sup}}=\sup_{x\in X}A(x)$.

A mapping $\lhd : I\times I\to I$ is called a *fuzzy implication* on I if it satisfies the boundary conditions according to the Boolean implication, and is decreasing in the first and increasing in the second argument. Several classes of fuzzy implications have been studied in the literatures [2,31]. The definitions of three main classes of these operations are recalled as follows.

Let $\top$, $\bot$ and N be a t-norm, a t-conorm and an involutive negator, respectively. A fuzzy implication $\lhd$ is called

- an *S-implication* based on $\bot$ and N if $a\lhd b=N(a)\bot b$ for all $a,b\in I$;
- an *R-implication* based on $\top$ if $a\lhd b=\sup\{c\in I \mid a\top c\leqslant b\}$ for all $a,b\in I$;
- a *QL-implication* based on $\top$, $\bot$ and N if $a\lhd b=N(a)\bot(a\top b)$ for all $a,b\in I$.