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A survey on the categorical term construction with applications

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Abstract

This paper gives a survey on the categorical term construction based on the free algebra algorithm. In the framework of monoidal biclosed and cocomplete categories a possible concept of signature for finitary theories is introduced. Applications of these constructions are given in Goguen's category and in the category of complete lattices and join preserving maps.

Keywords: Functor algebras; Term monads; Quantale; Algebra; Category theory

1. Motivation

The literature in computing science and mathematics seems to rely almost universally on non-categorical definitions for terms. Terms are defined usually as elements of a set, for example in the Zermelo–Fraenkel set theory (ZFC), with heavy use of natural language. This is astonishing, since categorical term construction is known for quite a long time. The main idea of the paper is to give a survey and some examples in specific categories.

Indeed, the typical situation is as follows: Operator symbols of arity n ($n \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of natural numbers including 0) are seen as elements of a so-called operator set Ω_n and variables are seen as elements of a variable set X. Signature Σ is then given as a system of operator sets, thus, $\Sigma = (\Omega_n)_{n \in I}$, where I is a non-empty subset of $\mathbb{N}_0$. Terms are seen as elements of a terms set $T_{\Sigma(X)}$, where $\Sigma(X)$ indicates that the variable set X is associated to the signature Σ. The production of the terms set from operators and variables is described using natural language and ZFC with its first-order metalanguage. Clearly, it might be sometimes sufficient to define, study and describe some systems, like 'logic' including a terms set, using these metalanguages only. However, the use of these metalanguages gives only a little insight or "guidelines" on incorporating uncertainty to terms as there is uncertainty incorporated in prior to operators and variables, for example. Naturally, there are different kinds of possibilities to consider generalized terms, not only to incorporate uncertainty. How to justify the generalization in hand? Clearly,

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the answer depends on the preciseness and "richness" of the used metalanguages, the languages by which the current system is defined and studied externally.

The situation is complicated when variables and n-ary operators are considered to be of some given sort (or type). Definitions for signatures, variables and terms set in many-sorted environment can be found in many textbooks, for example [21]. In the following we shortly summarize these definitions in the case of finitary theories:

Signature: A signature is a pair $\Sigma = (S, \Omega)$, where S is a set of sorts and Ω is a set of operator (symbols) $\omega \colon \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$, where $\mathtt{s}_1, \ldots, \mathtt{s}_n, \mathtt{s} \in S$ ($n \in \mathbb{N}, n \geq 1$). When $n = 0$ we write $\omega \colon \to \mathtt{s}$ and call these operators as constants. The arity of the operator $\omega \colon \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$ is $\mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$, and the arity of $\omega \colon \to \mathtt{s}$ is $\to \mathtt{s}$. One may write ω instead of $\omega \colon \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$ or $\omega \colon \to \mathtt{s}$ if no ambiguities arise. The sorts $\mathtt{s}_1, \ldots, \mathtt{s}_n$ are called argument sorts of the operator ω and $\mathtt{s}$ is called its target sort.

Variable: When a signature $\Sigma = (S, \Omega)$ is associated with a family $X = (X_\mathtt{s})_{\mathtt{s} \in S}$ of (disjoint) sets, we write $\Sigma(X)$, and each element of $X_\mathtt{s}$ is called variable of sort $\mathtt{s}$. It is assumed that variables in each $X_\mathtt{s}$ and operators in Ω are different.

Terms set: When a signature $\Sigma = (S, \Omega)$ and the family of variable sets $X = (X_\mathtt{s})_{\mathtt{s} \in S}$ are given, the terms set forming the family $T_{\Sigma(X)} = (T_{\Sigma(X),\mathtt{s}})_{\mathtt{s} \in S}$ is defined by simultaneous induction:

(i) $X_\mathtt{s} \subseteq T_{\Sigma(X),\mathtt{s}}$;
(ii) if $\omega \colon \to \mathtt{s} \in \Omega$, then $\omega \colon \to \mathtt{s} \in T_{\Sigma(X),\mathtt{s}}$;
(iii) if $\omega \colon \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s} \in \Omega$ and if $t_i \in T_{\Sigma(X),\mathtt{s}_i}$ for $1 \leq i \leq n$, then $\omega(t_1, \ldots, t_n) \in T_{\Sigma(X),\mathtt{s}}$.

Clearly, these informal definitions are in many cases sufficient and easily understood by readers. However, if someone had fuzzy sets $f \in [0,1]^X$ for variables and $g \in [0,1]^\Omega$ for operators then how to produce a fuzzy set of terms which depends on f and g? The above used metalanguage seems to be too poor for performance and justification of this kind of generalization. Moreover, variable substitution must be given informally. It is not possible to consider variable substitution basing on a function $\sigma \colon X \to T_{\Sigma(Y)}$ since X and $T_{\Sigma(Y)}$ are systems of sets, while one might consider a system of functions $\sigma = (\sigma_\mathtt{s})_{\mathtt{s} \in S}$, where $\sigma_\mathtt{s} \colon X_\mathtt{s} \to T_{\Sigma(X),\mathtt{s}}$ for all $\mathtt{s} \in S$. Indeed, the variable substitution must now be described by simultaneous induction, again.

On the other hand, in a categorical framework sets can reside in the underlying set theory, and also as objects in the category $\mathtt{Set}$ of sets. There is then a temptation to consider a functor which produces terms sets from operators and variables. Adámek presented a categorical term construction based on the free algebra algorithm [1]. Manes considered the production of, especially, one-sorted terms as a functor $\mathsf{T}_\Sigma \colon \mathtt{Set} \to \mathtt{Set}$, and he proved in [23] that the one-sorted term functor can be extended to a monad $\mathbf{T}_\Sigma$, however, the constructions of Manes are still treated relatively informally. The performance of substitution was given as morphisms in the Kleisli category $\mathtt{Set}_{\mathbf{T}_\Sigma}$. The first attempts to purely categorical constructions of the one-sorted term monad over $\mathtt{Set}$ appeared much later, e.g., in [13]. To incorporate fuzziness to terms was considered in [11,10] as a composition of term monad with the many-valued powerset monad $\mathbf{L}$ (see [23]) leading to performance of substitution in the Kleisli category $\mathtt{Set}_{\mathbf{L} \bullet \mathbf{T}_\Sigma}$. The first attempt to construct a term monad over a category of fuzzy sets, namely $\mathtt{Set}(\mathfrak{L})$, $\mathfrak{L}$ is a complete lattice, was given in [12], and a construction of a term monad over a sorted category of sets and sorted $\mathtt{Set}(\mathfrak{L})$ was initiated in [8]. The category $\mathtt{Set}(\mathfrak{L})$ was essentially determined in [15], thus, referred to as *Goguen's category* in this paper. A construction of a term monad over sorted monoidal biclosed categories was given in [9] with further notions and developments leading to variable substitutions in the corresponding Kleisli categories.

The paper is organized as follows. In Section 2 we recall certain categorical concepts for convenience of the readers. Especially, we recall monoidal categories, since they play a key role in this paper. In Section 3 we recall free functor algebras and derive term functors. Then, Section 4 discusses the many-sorted term construction in monoidal biclosed and cocomplete categories. In this context for the convenience of the reader we also recall the one-sorted term construction.

After these preparations we explain two special monoidal categories in Section 5. Subsection 5.1 recalls Goguen's category as a monoidal biclosed category, and Subsection 5.2 develops the monoidal structure of the category $\mathtt{Sup}$ of complete lattices and join preserving maps. As an illustration of the previous theory Section 6 discusses the construction of some important one-sorted or many-sorted algebras: In Subsection 6.1 we study magmas in Goguen's category,