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Fuzzy Sets and Systems 298 (2016) 194–206

FUZZY sets and systems

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Stochastic independence for probability MV-algebras

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Received 12 October 2014; received in revised form 4 August 2015; accepted 5 August 2015

Available online 20 August 2015

Abstract

We propose a notion of stochastic independence for probability MV-algebras, addressing an open problem posed by Riečan and Mundici. Furthermore, we prove a representation theorem for probability MV-algebras and we get MV-algebraic versions of the Hölder inequality and the Hausdorff moment problem.

Keywords: Probability MV-algebra; State; Stochastic independence

0. Introduction

MV-algebras were defined by Chang [3] and they stand to Łukasiewicz ∞-valued logic as boolean algebras stand to classical logic. The theory of MV-algebras is highlighted by Mundici's categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit (ℓu-groups) [25]. The twofold nature of MV-algebras, i.e. generalizations of boolean algebras and unit intervals of ℓu-groups, has the following probabilistic counterpart: *probability MV-algebras* are the main ingredient of the extension of Carathéodory boolean algebraic probability theory to many-valued events [27, Chapter 13], while *the (finitely additive) states* are in one-to-one correspondence with normalized states on ℓu-groups [27, Chapter 10].

In Section 2, using results from [21] and [22], we prove that any MV-algebra with a faithful state can be embedded in the interval $[\mathbf{0}, \mathbf{1}]$ of a suitable Banach lattice $L^1(\mu)$ (Theorem 2.1). As a direct consequence we get *the Hölder inequality* and *the Hausdorff moment problem* for *PMV-algebras* [8] (MV-algebras endowed with an internal product), and for *fMV-algebras* [20] (PMV-algebras which additionally have multiplication with scalars from [0, 1]).

A *probability MV-algebra* is a pair (A, s), where A is a σ-complete MV-algebra and s is a σ-continuous faithful state. Probability MV-algebras were defined by Riečan and Mundici in the comprehensive study [30], which also contains the following open problem:

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http://dx.doi.org/10.1016/j.fss.2015.08.008

"[...] Assuming M and N to be probability MV-algebras, generalize the classical theory of "stochastically independent" σ-subalgebras as defined in Fremlin's treatise [Measure Theory, 325L]."

This problem is addressed in [21] only for MV-algebras endowed with finitely additive states, but no solution is given for probability MV-algebras. In Section 3, inspired by [13, 253F], we propose a notion of stochastic independence for probability MV-algebras. We associate to any probability MV-algebras (A, s_A) and (B, s_B) a probability MV-algebra (T, s_T) and a biadditive function $\beta : A \times B \to T$ such that $s_T(\beta(a, b)) = s_A(a) \cdot s_B(b)$, for any $a \in A$ and $b \in B$. Moreover, (T, s_T) satisfies a universal property (see Theorem 3.1) and, consequently, its group reduct is unique up to isomorphism.

1. Preliminaries

1.1. Algebraic structures

An *MV-algebra* is an algebraic structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is a commutative monoid, * is an involution (i.e. $a^{**} = a$ for any $a \in A$) and the equations $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$ and $a \oplus 0^* = 0^*$ are satisfied for any $a, b \in A$ [3,4,27]. The variety of MV-algebras is generated by $([0, 1], \oplus, ^*, 0)$ where $a \oplus b = \min(a + b, 1)$ and $a^* = 1 - a$ for any $a, b \in [0, 1]$. **MV** is the category having MV-algebras as objects and homomorphisms of MV-algebras as morphisms.

One also defines the constant $1 = 0^*$, the operation $a \odot b = (a^* \oplus b^*)^*$ and the distance function $d(a, b) = (a \odot b^*) \oplus (b \odot a^*)$ for any $a, b \in A$. Setting $a \leq b$ if and only if $a^* \oplus b = 1$, then $(A, \leq, 0, 1)$ is a bounded distributive lattice such that $a \vee b = (a^* \oplus b)^* \oplus b$ and $a \wedge b = (a^* \vee b^*)^*$ for any $a, b \in A$. An MV-algebra A is σ-complete (Dedekind–MacNeille complete) if its lattice reduct is a σ-complete (Dedekind–MacNeille complete) lattice.

If A is an MV-algebra we define a partial operation $+$ as follows: for any $a, b \in A$, $a + b$ is defined if and only if $a \leq b^*$ and, in this case, $a + b = a \oplus b$. This operation is cancellative and any MV-algebra A satisfies the Riesz decomposition property [9, Section 2.9]. Throughout the paper we use the following notation:

$$na = \underbrace{a + \cdots + a}_{n} \quad \text{and} \quad n_{\oplus}a = \underbrace{a \oplus \cdots \oplus a}_{n}$$

where $a \in A$ and $n \geq 1$ is a natural number.

If A and B are MV-algebras then a function $\omega : A \to B$ is *additive* if $f(a + b) = f(a) + f(b)$ whenever $a \leq b^*$. Biadditive functions are functions $\omega : A \times B \to C$ such that $\omega(\cdot, b)$ and $\omega(a, \cdot)$ are additive for any $a \in A$ and $b \in B$. A bimorphism is a biadditive function that is $\vee$-preserving and $\wedge$-preserving in each component. We refer to [12] for basic results on additive functions.

An *ideal* in A is a lower subset I that contains 0 and it is closed under $\oplus$. A maximal ideal is an ideal that is maximal in the set of all ideals ordered by set-theoretic inclusion. A *semisimple MV-algebra* is an MV-algebra in which the intersection of all maximal ideals is $\{0\}$. Equivalently, an MV-algebra A is semisimple if and only if there exists a compact Hausdorff space X such that A can be embedded in the MV-algebra $C(X) = \{f : X \to [0, 1] \mid f \text{ continuous}\}$ with pointwise operations [4, Corollary 3.6.8].

An *ℓ-group* is an abelian group that is also a lattice and the following holds

$$x \leq y \text{ implies } x + z \leq y + z \quad \text{for any } x, y, z \in G.$$

If G is an ℓ-group, an element $u \in G$ is a (*strong order*) *unit* if $u \geq 0$, and for every $x \in G$ there is a natural number $n \geq 1$ such that $nu \geq (x \vee -x)$. An ℓ-group is *unital* (for short, it is an ℓu-group) if it is endowed with a distinguished strong unit. The category of unital ℓ-groups and unit-preserving homomorphisms of ℓ-groups is denoted **auG**.

If (G, u) is a unital ℓ-group we denote $[0, u] = \{x \in G | 0 \leq x \leq u\}$ and we define

$$x \oplus y = (x + y) \wedge u \quad \text{and} \quad \neg x = u - x \quad \text{for any } x, y \in [0, u].$$

Then $[0, u]_G = ([0, u], \oplus, \neg, 0)$ is an MV-algebra. For any MV-algebra A there exists an ℓu-group (G, u) such that $A \simeq [0, 1]_G$. Moreover, the following property holds: for any $x \geq 0$ in G there exist a natural number $n \geq 1$ and $a_1, \ldots, a_n \in A$ such that $x = a_1 + \cdots + a_n$.