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The role of metalanguage in graded logical approaches

Soma Dutta [a,*], Mihir K. Chakraborty [b]

[a] Institute of Mathematics, University of Warsaw, Poland
[b] Jadavpur University, Kolkata, India

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Abstract

This paper is an attempt to show how the metalanguage is important in building a logic dealing with uncertainties. We shall address this issue from two different angles. In one we shall propose a new way to look at the notion of consequence by introducing a series of metalogical notions based on the metalanguage and its interpretation; and in the other we shall present concrete systems of graded logic, which are generated based on both the object language, metalanguage, and their interrelations.

Keywords: Fuzzy consequence; Implicative consequence; Graded consequence; Metalogic; Consistency-generating relation

1. Introduction

What do we understand by the graded logical approach? Is it just the endorsement that in a logical system grades other than the top and the least of a typical lattice structure may be assigned to its formulae, or both to the formulae and the reasoning mechanism? More generally, is this all about focusing on a graded object language keeping its metalanguage two-valued and classical, or keeping the room open for a situation where many-valuedness of object level can naturally be carried over to the metalinguistic concepts? The approaches dealing with the former point of view are usually known as many-valued logics and/or fuzzy logics. The latter is the basic philosophical stance of the theory of graded consequence (GCT). GCT proposes a set-up where a logical system is graded not only because of the object language but its metatheory too is in general graded.

The purpose of this paper is to focus on the essential role played by the metalanguage and metalogic in logical discourse. Usually in logic-studies the major emphasis is laid on the object language and the logical connectives present in it. But in fact, the discourse cannot take place without a metatheory. This becomes evident while discussing many-valued or fuzzy logics. But even in these logics no explicit mention of metalogic is usually made. Theory of graded consequence brings this point to the fore by showing a clear distinction between the levels [8].

* Corresponding author.
E-mail addresses: somadutta9@gmail.com (S. Dutta), mihirc4@gmail.com (M.K. Chakraborty).

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There are some subtle distinctions between many-valued logics and fuzzy logics. With the publication of the paper viz. 'The logic of inexact concepts' by Goguen [15] in the year 1968–1969, fuzzy logic emerged as a discipline in logic. In 1975 based on the theory of fuzzy sets Zadeh proposed the idea of approximate reasoning [27] as a mathematical model of human reasoning. The word 'Fuzzy logic', then, was being used in a broad sense. Gradually, through the works [6,16,17,19,21] and many others the idea of fuzzy logic started to get a shape in a more strict sense where the use of fuzzy set theory alone does not determine the realm of fuzzy logic. Later, this branch of mathematical logic based on fuzzy set theory became familiar in the name of FLn, fuzzy logic in the narrow sense [17]. In Hàjek's [17] term fuzzy logic is a system, endowed with the ability of *deriving partially true (graded) conclusion from partially true (graded) premises*. It is to be noted that *derivation* itself is not a graded concept here. Zadeh has differentiated fuzzy logic from many-valued logics in the following sense [17]: "…*fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic*". Here too, as pointed out by Pelta [22], there is no notion of multivalence in the concept of 'inferencing': "*Until now the construction of superficial many-valued logics, that is, logics with an arbitrary number (bigger than two) of truth values but always incorporating a binary consequence relation, has prevailed in investigations of logical many-valuedness.*"

The same concern was echoed in the following lines of Parikh [20], where he mentioned: "…*we seem to have come no closer to observationality by moving from two valued logic to real valued, fuzzy logic. A possible solution …is to use continuous valued logic not only for the object language but also for the metalanguage*." And Zadeh's extended fuzzy logic [27] also could be counted as an account of the same concern.

The formal mathematical set-up of GCT provides a general framework for the metatheory of a logic where *the procedure of deriving partially true conclusion from a set of partially true premises is itself also a matter of degree or grade*. GCT meticulously takes care of the following points.

(i) Both object and metalevel concepts have their own respective linguistic framework, and interpretations; neither these languages nor the algebraic structures for their interpretations are necessarily the same.
(ii) Like object language formulae, metalevel sentences involving metalinguistic connectives and quantifiers are also truth functional; they depend on the interpretations of the metalinguistic entities.
(iii) A particular logic depends on both the object language and metalanguage, their interpretations, and interrelations.

We shall show how the role of the metalinguistic concepts and their interpretation make a difference in building a logic. In this regard, we shall first discuss the relationships of the notion of graded consequence with other existing notions of consequence in fuzzy context in which attempts are made to some extent to incorporate gradedness in the notion of consequence. We shall also propose a different perspective to interpret some concepts involved in these notions of consequences. Another important part of the paper is the idea of building logics in the framework of GCT by integrating two logics at the object and metalevels.

2. An overview of various notions of consequence in fuzzy context

In classical context, the notions of consequence operator [26] and consequence relation [13] are equivalent in the sense that considering one as the primitive notion the other can be obtained. In fuzzy logical set-up, Pavelka [21] proposed the notion of fuzzy consequence operator, also called fuzzy closure operator, generalizing the notion of consequence in the sense of Tarski [26]. A fuzzy consequence operator is proposed to be a function C from the set of all fuzzy sets over formulae to itself, i.e., $C\colon \mathcal{F}(F) \mapsto \mathcal{F}(F)$, satisfying

($C1$) $X \subseteq C(X)$,
($C2$) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$, and
($C3$) $C(C(X)) = C(X)$, where $X \subseteq Y$ stands for $X(\alpha) \leq Y(\alpha)$ for all $\alpha \in F$, representing the notion of inclusion in fuzzy context.

Chakraborty [2,3], on the other hand, introduced the notion of graded consequence to generalize the notion of consequence relation [13] in fuzzy context. A graded consequence relation is a fuzzy relation $|\!\sim$ between P(F), the set of all sets of formulae, and F, the set of all formulae, satisfying the following conditions.