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Convex combinations of fuzzy logical operations

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Abstract

One may ask when (non-trivial) convex combinations of fuzzy logical operations result in operations of the same type. This question has been intensively studied for triangular norms and it still remains open for continuous ones. An equivalent problem is obtained for triangular conorms. We show equivalence with the analogous problem for S-implications. Negative answers are obtained for strong fuzzy negations and for R-implications corresponding to continuous triangular norms. The situation for Q-implications and related problems are discussed.

Keywords: Fuzzy implication; R-implication; S-implication; Q-implication; Convex combination; Triangular norm; Triangular conorm; Fuzzy negation

1. Introduction

The paper follows and continues the research done mainly in [28,32], as outlined in the abstract (see below for details). Only operations derived from continuous Archimedean triangular norms and conorms were considered in [28], but here we proceed without this assumption. We concentrate on continuous operations as they seem the most important from the point of view of applications. The continuity assumption also discards some counterexamples (cf. [11]).[1]

In Section 2, we recall the basic definitions. Readers familiar with basics of fuzzy logical operations may skip it and return to particular definitions only in case of hesitations. Section 3 is devoted to the problem of convex combinations of t-norms and several of its weakened formulations. In Section 4, we summarize main tools used in the sequel and apply them directly to some questions. The rest of the paper is devoted to a discussion on convex combinations of particular types of fuzzy implications.

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[1] We deal also with R-implications, which usually are not continuous, even if they are residua of continuous triangular norms.

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2. Basic definitions

A *fuzzy negation* is a unary operation $N: [0,1] \to [0,1]$ which is involutive and non-increasing, i.e., $N(N(x)) = x$ and $N(x) \geq N(y)$ for all $x, y \in [0,1]$ such that $x \leq y$. In some papers and monographs (e.g. [13]), more general negations are allowed; then our notion defines so-called *strong* or *involutive* fuzzy negation. We shall deal only with involutive fuzzy negations.

A *triangular norm* (a *t-norm*) $T: [0,1]^2 \to [0,1]$ is a commutative, associative, non-decreasing binary operation such that $T(x,1) = x$ for all $x \in [0,1]$ (see [3,12,17,37]). Dually, a *triangular conorm* (a *t-conorm*) $S: [0,1]^2 \to [0,1]$ is a commutative, associative, non-decreasing binary operation such that $S(x,0) = x$ for all $x \in [0,1]$. In this paper, we shall deal only with continuous t-norms and t-conorms. A continuous t-norm T, resp. a t-conorm S, is *Archimedean* if, for each $x \in]0,1[$, $T(x,x) < x$, resp. $S(x,x) > x$.[2] Continuous Archimedean t-norms and t-conorms are called *strict* if they are strictly increasing on the open unit square $]0,1[^2$; otherwise, they are called *nilpotent* [3,17,37].

Let T be a continuous Archimedean t-norm. A *multiplicative generator* of T is a strictly increasing function $\theta: [0,1] \to [0,1]$ such that $\theta(1) = 1$ and

$$T(x,y) = \begin{cases} \theta^{-1}(\theta(x)\cdot\theta(y)) & \text{if } \theta(x)\cdot\theta(y) \geq \theta(0), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

An *additive generator* of T is a continuous, strictly decreasing function $t: [0,1] \to [0,\infty]$ such that $t(1) = 0$ and

$$T(x,y) = \begin{cases} t^{-1}(t(x)+t(y)) & \text{if } t(x)+t(y) \leq t(0), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Every continuous Archimedean t-norm has (non-unique) multiplicative and additive generators [1,19,24].

If $(T_\alpha)_{\alpha\in A}$ is a family of t-norms and $((a_\alpha, b_\alpha))_{\alpha\in A}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the following function $T: [0,1]^2 \to [0,1]$ is a t-norm [10]:

$$T(x,y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha)\cdot T_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, b_\alpha[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

The t-norm T is called the *ordinal sum* of $(T_\alpha)_{\alpha\in A}$. As a consequence of [24], a t-norm is continuous if and only if it is (uniquely) representable as an ordinal sum of continuous Archimedean t-norms.

There is no entirely satisfactory and generally accepted definition of a fuzzy implication. We mostly work with the following one:

Definition 1. (See [4,35].) A function $I: [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

(I1) $I(0,0) = I(1,1) = I(0,1) = 1$, $I(1,0) = 0$,
(I2) I is non-decreasing in the second variable,
(I3) I is non-increasing in the first variable.

Condition (I1) is indispensable, it says that I extends the classical (Boolean) implication. The remaining two monotonicity conditions are natural; nevertheless, they are not always required, e.g., in [18]. The conditions of Definition 1 are rather weak, so we shall study the following specific fuzzy implications [18,29]:

- *R-implications* of the form

  $$I_T(x,y) = \max\{z \in [0,1] \mid T(x,z) \leq y\},$$

  where T is a continuous t-norm (it is also called a *residuated implication* or the *residuum* of T),

[2] Also non-continuous Archimedean t-norms and t-conorms are defined, see [17]. The conditions used here are simpler and equivalent in case of continuous t-norms and t-conorms.