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Short communication

Lattice representations of interval-valued fuzzy sets

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Abstract

In this paper, a solution of the problem of the interval-valued fuzzy sets synthesis, is given. The codomain is considered to be a lattice of intervals. The necessary and sufficient conditions for a family of subsets to be a family of cut sets of an interval-valued fuzzy set are given.

Keywords: Interval-valued fuzzy sets; Lattice-valued fuzzy sets; Cuts

1. Introduction

An interval-valued fuzzy set on a universe X is a mapping from X to the set of all closed sub-intervals of the real interval [0, 1]. This type of fuzzy sets has been intensively investigated, not only its theoretical aspects, but also its numerous applications.

It's a well known fact that every fuzzy structure is determined by the family of subsets of the universe X, i.e. by the family of its cut sets. Many properties of fuzzy structures can be investigated due to their transferability to cut sets. These properties of fuzzy structures are named "cutworthy" properties by Klir and Yuan [6]. Kerre [5] has pointed out: "…everytime when a crisp notion has been fuzzified the question about such a characterization[1] arises and also vice versa: under which conditions can a fuzzy structure be reconstructed from the knowledge of its α- levels". Therefore, the relevant question is: under which conditions can an interval-valued fuzzy set be reconstructed from the family of its cut sets.

Here, we refer to certain publications that are connected with the aspects we are working with – cuts and representation theorem by cuts: [4,9–12] for fuzzy sets, [2,7,8,15,16] for L-fuzzy sets, [13,14,18] for poset-valued fuzzy sets. The concept of cut sets of interval-valued fuzzy sets was introduced by Zeng and Shi [23]. They introduced eight cut sets for interval-valued fuzzy sets, investigated their properties and proposed three decomposition theorems based on introduced cuts. In the next step, Zeng, Shi and Li [24] introduced generalized cut sets and named them 1–1 type of interval-valued nested set and 1–2 type of interval-valued nested set. They proposed two representation theorems based on those cuts.

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[1] Via the α- level (cut) sets.

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Yuan, Li and Lee have introduced new kinds of cuts [20–22] which are three-valued fuzzy sets. They established representation theorems for interval-valued fuzzy sets [19] by those cuts.

There are some common points in our work and the work of Zeng, Shi and Li [24], but our approach is basically different. We have considered a given family of subsets of X and given necessary and sufficient conditions for the existence of interval-valued fuzzy set with the family of cuts equal to the family we started with. Basically, we have investigated the properties that the given family of sets must satisfy and our most important request was for the family of cuts to be equal to that family, which is not the case in the above mentioned papers.

Paper [2] solves the general problem of synthesis for lattice-valued fuzzy sets, which is formulated as follows:

Let L be a fixed complete lattice. Characterize under which condition $\mathcal{F} \subseteq \mathcal{P}(X)$ is a collection of cut sets of a fuzzy set $\mu : X \to L$.

This paper proves that the general form of lattice-valued fuzzy sets (considering families of cuts) is the type of fuzzy sets having the codomain $\{0, 1\}^c$ for a suitably chosen cardinal c (or Boolean-valued fuzzy sets).

The techniques developed in this paper are used here to prove the theorem of synthesis for interval-valued fuzzy sets.

2. Preliminaries

An interval-valued fuzzy set [3] (IVFS) on a universe X is a mapping $\mu : X \to Int([0, 1])$, where $Int([0, 1])$ is a family of all closed sub-intervals of $[0, 1]$. The family of all interval-valued fuzzy sets on X is denoted by $\mathcal{IVFS}(X)$.

We apply the usual denotation such that for every $\mathbf{a} \in Int([0, 1])$ the lower and upper limits of the interval are denoted by $\underline{a}$ and $\bar{a}$, respectively, i.e., $\mathbf{a} = [\underline{a}, \bar{a}]$.

We consider interval-valued fuzzy sets to be a special type of lattice-valued fuzzy sets, since $Int([0, 1])$ can be considered a lattice under different orderings. We consider $Int([0, 1])$ a lattice denoted by $\mathcal{L}^*$ under the ordering $\leqslant$ defined below.

We define $\mathcal{L}^* = (L^*, \leqslant)$ as follows [1]:

$$L^* = \left\{[\underline{x}, \bar{x}] \mid (\underline{x}, \bar{x}) \in [0, 1]^2, \underline{x} \leqslant \bar{x}\right\} \tag{1}$$

with the ordering $[\underline{x}, \bar{x}] \leqslant [\underline{y}, \bar{y}]$ iff $\underline{x} \leqslant \underline{y}$ and $\bar{x} \leqslant \bar{y}$ for all $[\underline{x}, \bar{x}], [\underline{y}, \bar{y}] \in L^*$.

As usual, $\geqslant$ is defined to be the inverse relation to $\leqslant$.

It is easy to see that $(L^*, \leqslant)$ is a complete lattice with the bottom element $\mathbf{0} = [0, 0]$ and the top element $\mathbf{1} = [1, 1]$ with the infimum and the supremum (respectively) for a family $[\underline{x_i}, \overline{x_i}]$ given by

$$\bigwedge_{i\in I}[\underline{x_i}, \overline{x_i}] = \left[\inf_{i\in I} \underline{x_i}, \inf_{i\in I} \overline{x_i}\right] \quad \text{and} \quad \bigvee_{i\in I}[\underline{x_i}, \overline{x_i}] = \left[\sup_{i\in I} \underline{x_i}, \sup_{i\in I} \overline{x_i}\right].$$

Let $\mu : X \to L^*$ be an interval-valued fuzzy set. Then, for $\mathbf{p} = [\underline{p}, \bar{p}] \in L^*$, a cut set is defined by: $\mu_{\mathbf{p}} = \{x \in X \mid \mu(x) \geqslant [\underline{p}, \bar{p}]\}$.

Let $\mathbf{n}$ and $\mathbf{m}$ be finite chains (totally ordered sets) with n and m elements, respectively. Then, we consider the direct product of two chains $\mathbf{n} \times \mathbf{m}$, which is a set of all ordered couples (x, y), such that $x \in \mathbf{n}$ and $y \in \mathbf{m}$.

The operations $\vee$ and $\wedge$ are defined as maximum and minimum componentwise: for all $x_1, y_1 \in \mathbf{n}$ and $x_2, y_2 \in \mathbf{m}$,

$$(x_1, x_2) \wedge (y_1, y_2) = \big(\min(x_1, y_1), \min(x_2, y_2)\big),$$
$$(x_1, x_2) \vee (y_1, y_2) = \big(\max(x_1, y_1), \max(x_2, y_2)\big).$$

With $\mathbf{n} \times \mathbf{m} \oplus \mathbf{1}$ we denote the lattice $\mathbf{n} \times \mathbf{m}$ with the top element $\mathbf{1}$ added.

A *closure operator* on a lattice $\mathcal{L}$ is a function $C : L \to L$, satisfying: $p \leqslant C(p)$; $p \leqslant q \to C(p) \leqslant C(q)$ and $C(C(p)) = C(p)$. If $p = C(p)$, then p is a *closed element* under the corresponding closure operator. The set of all closed elements is closed under infima in the lattice L and the top element of L is always a closed element.

The following theorem is a solution of a well known problem of the lattice-valued fuzzy sets synthesis:

Theorem 1. *(See [16].) Let F be a family of subsets of a nonempty set X which is closed under intersections and contains X. Let $\mu : X \to F$ be defined by*

$$\mu(x) = \bigcap(p \in F \mid x \in p).$$