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On S-homogeneity property of seminormed fuzzy integral: An answer to an open problem

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ABSTRACT

We give an answer to Problem 9.3 stated by Mesiar and Stupňanová [8]. We show that the class of semicopulas solving this problem contains any associative semicopula S such that for each $a \in [0, 1]$ the function $x \mapsto \mathrm{S}(a, x)$ is continuous and increasing on a countable number of intervals.

1. Introduction

Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of a non-empty set X, and let $\mathcal{S}$ be the family of all measurable spaces. The class of all $\mathcal{A}$-measurable functions $f : X \to [0, 1]$ is denoted by $\mathcal{F}_{(X,\mathcal{A})}$. A *capacity* on $\mathcal{A}$ is a non-decreasing set function $\mu : \mathcal{A} \to [0, 1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$. We denote by $\mathcal{M}_{(X,\mathcal{A})}$ the class of all capacities on $\mathcal{A}$.

Suppose that $\mathrm{S} : [0, 1]^2 \to [0, 1]$ is a non-decreasing function in both coordinates with neutral element 1, called a *semicopula*, a *conjunctor* or a *t-seminorm* (see [2,3]). It is clear that $\mathrm{S}(x, y) \leqslant x \wedge y$ and $\mathrm{S}(x, 0) = 0 = \mathrm{S}(0, x)$ for all $x, y \in [0, 1]$. We denote the class of all semicopulas by $\mathfrak{S}$. Typical examples of semicopulas include: $\mathrm{M}(a, b) = a \wedge b$, $\Pi(a, b) = ab$, $\mathrm{S}(x, y) = xy(x \vee y)$ and $\mathrm{S}_L(a, b) = (a + b - 1) \vee 0$; S_L is called the *Łukasiewicz t-norm* [6]. Hereafter, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

The generalized Sugeno integral is defined by

$$\mathbf{I}_{\mathrm{S}}(\mu, f) := \sup_{t \in [0,1]} \mathrm{S}(t, \mu(\{f \geqslant t\})),$$

where $\{f \geqslant t\} = \{x \in X : f(x) \geqslant t\}$, $(X, \mathcal{A}) \in \mathcal{S}$ and $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$. In the literature, $\mathbf{I}_{\mathrm{S}}$ is also called the *seminormed fuzzy integral* [4,7,9]. Replacing semicopula S with M, we get the *Sugeno integral* [11]. Moreover, if $\mathrm{S} = \Pi$, then $\mathbf{I}_{\Pi}$ is called the *Shilkret integral* [10].

Below we present Problem 9.3 from [8], which was posed by Hutník during *The Twelfth International Conference on Fuzzy Set Theory and Applications* held from January 26 to January 31, 2014 in Liptovský Ján, Slovakia.

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Problem 9.3 *To characterize a class of semicopulas* S *for which the property*

$$(\forall_{a\in[0,1]}) \quad \mathbf{I}_{\mathrm{S}}(\mu, \mathrm{S}(a, f)) = \mathrm{S}(a, \mathbf{I}_{\mathrm{S}}(\mu, f)) \tag{1}$$

holds for all $(X, \mathcal{A}) \in \mathcal{S}$ *and all* $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$.

Hutník et al. [5,8] conjectured that (1) characterizes the two element class $\{\mathrm{M}, \Pi\}$. We show that (1) holds for any associative semicopula with continuous selections satisfing some mild conditions.

2. Main results

Let $\mathfrak{S}_0$ denote the set of all semicopulas S which fulfill the following two conditions:

(C1) S is associative, i.e. $\mathrm{S}(\mathrm{S}(x, y), z) = \mathrm{S}(x, \mathrm{S}(y, z))$ for all $x, y, z \in [0, 1]$,
(C2) $[0, 1] \ni x \mapsto \mathrm{S}(a, x)$ is continuous for each $a \in (0, 1)$.

The class $\mathfrak{S}_0$ is non-empty as $\mathrm{M}, \Pi, \mathrm{S}_L \in \mathfrak{S}_0$. If we additionally assume that the function $[0, 1] \ni a \mapsto \mathrm{S}(a, x)$ is continuous for each $x \in (0, 1)$, then S is a continuous t-norm (see [1], Corollary 2.4.4, or [6], Theorem 2.43). It is an open problem whether $\mathfrak{S}_0$ contains only continuous t-norms.

We prove that the property (1) implies that $\mathrm{S} \in \mathfrak{S}_0$.

Theorem 2.1. *If (1) holds for all* $(X, \mathcal{A}) \in \mathcal{S}$ *and all* $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$, *then* $\mathrm{S} \in \mathfrak{S}_0$.

Proof. The equality (1) is obvious for $a \in \{0, 1\}$, so we assume that $a \in (0, 1)$. First, we show that (1) implies that S is an associative semicopula. Indeed, put $f = b\mathbb{1}_A$ in (1), where $b \in [0, 1]$ and $A \in \mathcal{A}$. Then (1) takes the form

$$\sup_{t\in[0,1]} \mathrm{S}(t, \mu(A \cap \{\mathrm{S}(a, b) \geqslant t\})) = \mathrm{S}\Big(a, \sup_{t\in[0,1]} \mathrm{S}(t, \mu(A \cap \{b \geqslant t\}))\Big).$$

Clearly, $\{\mathrm{S}(a, b) \geqslant t\} = X$ and $\{b \geqslant t\} = X$ for $t \in [0, \mathrm{S}(a, b)]$ and $t \in [0, b]$, respectively. Otherwise, both sets are empty. Hence

$$\sup_{t\in[0,\mathrm{S}(a,b)]} \mathrm{S}(t, \mu(A)) = \mathrm{S}\Big(a, \sup_{t\in[0,b]} \mathrm{S}(t, \mu(A))\Big).$$

Since S is non-decreasing, we get

$$\mathrm{S}(\mathrm{S}(a, b), c) = \mathrm{S}(a, \mathrm{S}(b, c)) \tag{2}$$

for all $a \in (0, 1)$ and $b, c \in [0, 1]$. The equality (2) holds also for $a \in \{0, 1\}$, so S is associative.

Second, we prove that $(C2)$ follows from (1) or, equivalently, that the following conditions are satisfied:

(C2a) $x \mapsto \mathrm{S}(a, x)$ is right-continuous for each $a \in (0, 1)$,
(C2b) $x \mapsto \mathrm{S}(a, x)$ is left-continuous for each $a \in (0, 1)$.

Denote by L and P the left-hand side and the right-hand side of Eq. (1), respectively. Let $X = [0, 1]$. Putting $\mu(A) = 0$ for $A \neq X$ yields

$$\mathrm{L} = \sup_{t\in[0,\, \inf_x \mathrm{S}(a,f(x))]} \mathrm{S}(t, 1) = \inf_{x\in[0,1]} \mathrm{S}(a, f(x)),$$

$$\mathrm{P} = \mathrm{S}\Big(a, \sup_{z\in[0,\, \inf_x f(x)]} \mathrm{S}(z, 1)\Big) = \mathrm{S}\Big(a, \inf_{x\in[0,1]} f(x)\Big).$$

Let $b_n \searrow b$ for a fixed $b \in [0, 1)$ and $f(x) = b_n \mathbb{1}_{[\frac{1}{n+1}, \frac{1}{n})}(x)$ for $x \in (0, 1)$ with $f(0) = f(1) = 1$. Hereafter, $a_n \searrow a$ means that $\lim_{n\to\infty} a_n = a$ and $a_n > a_{n+1}$ for all n. Since $\mathrm{L} = \mathrm{P}$, $\mathrm{P} = \mathrm{S}(a, b)$ and

$$\mathrm{L} = \inf_{x\in[0,1]} \mathrm{S}\big(a, f(x)\big) = \lim_{n\to\infty} \mathrm{S}(a, b_n),$$

the condition $(C2a)$ is satisfied. Now we show that $(C2b)$ is fulfilled. Set $\mu(A) = 1$ for all $A \neq \emptyset$. Obviously,

$$\mathrm{L} = \sup_{t\in[0,\, \sup_x \mathrm{S}(a,f(x))]} \mathrm{S}(t, 1) = \sup_{x\in[0,1]} \mathrm{S}(a, f(x)), \quad \mathrm{P} = \mathrm{S}\Big(a, \sup_{x\in[0,1]} f(x)\Big).$$

Let $b_n \nearrow b$ for some $b \in (0, 1]$, $f(x) = b_n \mathbb{1}_{[\frac{1}{n+1}, \frac{1}{n})}(x)$ for $x \in (0, 1)$ and $f(0) = f(1) = 0$. Since $\mathrm{L} = \mathrm{P}$, $\mathrm{P} = \mathrm{S}(a, b)$ and $\mathrm{L} = \sup_{x\in[0,1]} \mathrm{S}(a, f(x)) = \lim_{n\to\infty} \mathrm{S}(a, b_n)$, we obtain the condition $(C2b)$. $\square$

Next, we show that under an additional assumption on S, the condition $\mathrm{S} \in \mathfrak{S}_0$ is necessary and sufficient for (1) to hold. To prove our result we need the following lemma.

Lemma 2.1. *Suppose* $g, h : [0, 1] \to [0, 1]$ *and g is non-decreasing.*

(a) *If g is right-continuous, then* $g(\inf_{x\in[0,1]} h(x)) = \inf_{x\in[0,1]} g(h(x))$.
(b) *If g is left-continuous, then* $g(\sup_{x\in[0,1]} h(x)) = \sup_{x\in[0,1]} g(h(x))$.