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A technical note on two inconsistency indices for preference relations: A case of functional relation

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ABSTRACT

Different representations of valued preferences have been studied and used over the years; however, one common factor has been the prominence of the concept of consistency. Inconsistency indices have been introduced to estimate the deviation of preferences from a fully consistent form. In this note we shall recall two types of preference relations (reciprocal relations and multiplicative preference relations) and show that two inconsistency indices introduced in these two different frameworks are functionally related. Besides this main result, some reflections on the consequences of being functionally related are presented.

1. Introduction

In many decision making problems it is common practice to pairwise compare alternatives before obtaining a vector of priorities. Perhaps the foremost mathematical model using pairwise comparisons, but by no means the only one, is the Analytic Hierarchy Process by Saaty [22]. In these mathematical models, when a decision maker pairwise compares alternatives he/she makes an implicit use of the concept of (valued) preference relations. According to Alonso et al. [1], given a non-empty set of alternatives $X = \{x_1, \dots, x_n\}$ a *preference relation* P on X is characterized by a function $\mu_P \colon X \times X \to D$, where the ordered set D is the domain of representation of preference degrees. The choice of the domain D is the main difference among the most commonly used preference relations.

An important piece of information which becomes useful when using preference relations is the degree of inconsistency of the preferences. Namely, given a preference relation, we would like to associate to it a real number whose magnitude represents the irrationality of the preferences. Such indices can be seen as functions mapping preference relations into the real line. A wide number of inconsistency indices have been proposed; for example, the Consistency Index [22], the Geometric Consistency Index [2], the Harmonic Consistency Index [23], and the Cosine Consistency Index [20], just to name few. Comparative numerical [3] and axiomatic [5] studies on inconsistency indices are also present in the literature.

In this manuscript we briefly recall two types of preference relations – reciprocal relations and multiplicative preference relations – their interrelation, the definitions of inconsistency in their frameworks, and two prominent inconsistency indices developed in these two separate contexts. Then, the rest of this technical note will (i) show that these two seemingly different inconsistency indices are instead functionally related and (ii) discuss the importance and the consequences of finding that some indices are dependent to each other.

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2. Preliminaries

One special type of preference relation, called reciprocal relations is used when the domain of representation of the degrees of preference is the unit interval, i.e. $D = [0, 1]$. A *reciprocal relation* is represented by a matrix $\mathbf{R} = (r_{ij})_{n \times n}$ where $r_{ij} \in [0, 1]$ such that $r_{ii} = 0.5\, \forall i$ and $r_{ij} + r_{ji} = 1\, \forall i, j$. It is worth noting that, in the literature, reciprocal relations have often gone under the name ‘fuzzy preference relations’, although these latter ones were originally defined in a different way by Zadeh [26] and Orlovsky [28]. To avoid misunderstandings and terminological ambiguity, here we shall call them reciprocal relations, also to emphasize the reciprocity of preferences $r_{ij} + r_{ji} = 1\, \forall i, j$. A reciprocal relation is called (additively) *consistent* if [24],

$$(r_{ik} - 0.5) = (r_{ij} - 0.5) + (r_{jk} - 0.5) \quad \forall i, j, k \tag{1}$$

When satisfied, such a condition ensures that the preferences are non-contradictory and were expressed in a rational way. To formulate a measure of violation of the condition of consistency, Herrera-Viedma et al. [16,17] simplified formula (1) and reckoned that the theoretical value of the pairwise comparison r_{ik} obtained through indirect comparison via alternative x_j is $(r_{ij} + r_{jk} - 0.5)$. Then they concluded that

$$|\underbrace{r_{ij} + r_{jk} - 0.5}_{\text{estimation via } x_j} - r_{ik}|.$$

can be considered an estimation of the evaluation error. Accordingly, they considered arithmetic mean of the errors for all the $(n-2)$ x_j with $j \neq i, k$,

$$\frac{2}{3} \frac{1}{(n-2)} \sum_{\substack{1 \le j \le n \\ j \neq i,k}} |r_{ij} + r_{jk} - 0.5 - r_{ik}|,$$

where 2/3 is a rescaling factor. These values of the errors associated to entries r_{ik} can be aggregated to obtain the error associated to a given alternative x_i. This was done by averaging all the $(n-1)$ previous values for all $k \neq i$,

$$\frac{2}{3} \frac{1}{(n-2)} \frac{1}{(n-1)} \sum_{\substack{1 \le k \le n \\ k \neq i}} \sum_{\substack{1 \le j \le n \\ j \neq i,k}} |r_{ij} + r_{jk} - 0.5 - r_{ik}|.$$

Finally, these last quantities can be aggregated for all the n alternatives i to obtain an index of global inconsistency,

$$\frac{2}{3} \frac{1}{(n-2)} \frac{1}{(n-1)} \frac{1}{n} \sum_{1 \le i \le n} \sum_{\substack{1 \le k \le n \\ k \neq i}} \sum_{\substack{1 \le j \le n \\ j \neq i,k}} |r_{ij} + r_{jk} - 0.5 - r_{ik}|.$$

Ultimately, after some simplifications, this last quantity boils down to

$$IL(\mathbf{R}) = \frac{4}{n(n-1)(n-2)} \sum_{1 \le i < j < k \le n} |r_{ij} + r_{jk} - 0.5 - r_{ik}|. \tag{2}$$

where IL is the acronym of ‘inconsistency level’. Since $IL(\mathbf{R}) \in [0, 1]$, its reciprocal $cl = 1 - IL(\mathbf{R})$ has been similarly used to estimate consistency. However, for practical purposes they are equivalent, and for simplicity we shall restrict the analysis to IL, bearing in mind that the conclusions can be extended to cl. Note that the notation cl, which stands for consistency level, has been frequently used in the literature.

It is worth noting that $IL(\mathbf{R})$ has gained relevance in the literature and it has been used in approaches mixing consistency and consensus [6,12] and, as recalled by Ureña et al. [25], it has been employed by Zhang et al. [27] as an objective function in some linear programing models to estimate the values of missing comparisons. Its use in methods for estimating missing comparisons was already proposed by Alonso et al. [1] and studied by Chiclana et al. [13].

An alternative representation of preference relations is needed when the domain of representation D is the set of all positive real numbers. A *multiplicative preference relation* can be represented by a positive square matrix $\mathbf{A} = (a_{ij})_{n \times n}$ such that $a_{ij} = 1\, \forall i$ and $a_{ij} a_{ji} = 1\, \forall i, j$. For convenience, and behavioral and psychological considerations, the scale of values of entries a_{ij} is sometimes limited to the interval [1/9, 9]. Note that, in the literature, multiplicative preference relations are often called pairwise comparison matrices. In the case of multiplicative preference relations a condition of consistency was proposed in the following form [22],

$$a_{ik} = a_{ij} a_{jk} \quad \forall i, j, k. \tag{3}$$

Recently, the inconsistency index by Cavallo and D’Apuzzo [8,9] has attracted some attention. Such an index,

$$I_{CD}(\mathbf{A}) = \left(\prod_{1 \le i < j < k \le n} \max \left\{ \frac{a_{ik}}{a_{ij} a_{jk}}, \frac{a_{ij} a_{jk}}{a_{ik}} \right\} \right)^{\frac{1}{\binom{n}{3}}}, \tag{4}$$