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Notes on type-2 triangular norms and their residual operators

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ABSTRACT

Li recently investigated type-2 triangular norms and their residual operators. In this paper, the flaws in calculations of *T*-extension operations of t-(co)norms and their residual operations are presented with counterexamples. Moreover, incorrect conclusions are corrected.

1. Introduction

Mendel [10,17] pointed out that there are uncertainties in fuzzy logic systems. For example, the meaning of the words used in the antecedents and consequents of values can be uncertain, that is, words can mean different things to different people. However, ordinary fuzzy sets [21] cannot reflect these uncertainties, since they are characterized by crisp membership functions. Type-2 fuzzy sets were proposed by Zadeh [22] to model and manipulate these uncertainties, whose truth values are ordinary fuzzy sets on the unit interval and called fuzzy truth values. Type-2 fuzzy sets have been applied in many areas, such as decision making [1,25], control [4,15], pattern recognition [2,16] and clustering [5,9,16,23].

Type-2 t-norm is a favorite topic in the study of type-2 fuzzy sets. Gera and Dombi [6] studied the exact calculations of extended t-(co)norms on fuzzy truth values, which were further discussed in [13]. Starczewski [18] investigated extended t-norms on fuzzy truth interval or fuzzy truth numbers. Hu and Kwong [7] discussed t-norm extension operations. Moreover, Wang and Hu [20] studied the lattice structure of the algebra of fuzzy values. Based on it, fuzzy-valued t-norms and their residual operations were discussed on the algebra of fuzzy values, when t-norms are left continuous.

Recently, Li [14] investigated *T*-extension operations of t-(co)norms and their residual operators on the algebra of fuzzy truth values, where *T* denotes a t-norm. In [14], Li constructed type-2 t-norms on the algebra of fuzzy truth values with respect to the partial order $\leqslant$ and the partial order $\sqsubseteq$, respectively. Moreover, it was pointed out that extended minimum and its residual operation with respect to the partial order $\sqsubseteq$ form a BL-algebra on the algebra of convex normal and upper semicontinuous fuzzy truth values. Li also pointed out that the extended operation of a continuous Archimedean t-norm and its residual operation with respect to the partial order $\sqsubseteq$ form a BL-algebra on the family of decreasing, normal and upper semicontinuous fuzzy truth values. However, there are some flaws in [14]. For example, some calculations of *T*-extension of t-(co)norms and their residual operations are incorrect; the closedness of type-2 t-norms are unverified. Moreover, it is incorrect that extended minimum and the extension of a continuous Archimedean t-norm with their corresponding residual

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operations form BL-algebras. For the convenience of readers, we present the flaws mentioned above with counterexamples and provide the correct versions in our paper.

The content of the paper is organized as follows. In Section 2, we recall some fundamental concepts and related properties of t-norms and fuzzy truth values. In Section 3, we further investigate type-2 t-norms and their residual operations with respect to the ordinary partial order $\leqslant$ and give examples to show the flaws in [14]. Section 4 discusses type-2 t-norms and their residual operations with respect to the partial order $\sqsubseteq$. Moreover, we study the algebraic structures of type-2 t-norms and their residual operations with respect to the partial order $\sqsubseteq$. In the final section, we present some conclusions of our research.

2. Preliminaries

In this section, we present some basic concepts and terminology used throughout the paper.

A *fuzzy truth value* f is a mapping from the unit interval [0, 1] to itself. The family of all fuzzy truth values is denoted as $\mathscr{F}_2$. The operations on fuzzy truth values are defined as follows: for all $f, g \in \mathscr{F}_2$ and $x \in [0, 1]$,

$$(f \cap g)(x) = f(x) \wedge g(x),\ (f \cup g)(x) = f(x) \vee g(x),\ f^c(x) = (f(x))', \tag{1}$$

where $\wedge$, $\vee$, and $'$ are minimum, maximum and involutive negation on [0, 1], respectively. The order relation on fuzzy truth values is defined as $f \leqslant g \Leftrightarrow f(x) \leqslant g(x)$ for all $x \in [0, 1]$.

The algebra of fuzzy truth values $(\mathscr{F}_2, \sqcap, \sqcup, \widetilde{0}, \widetilde{1}, \sqsubseteq, \Subset)$ is investigated in [19], whereas

$$(f \sqcap g)(z) = \bigvee_{z=x\wedge y} \{f(x) \wedge g(y)\}, \qquad f \sqsubseteq g \text{ if } f \sqcap g = f,$$

$$(f \sqcup g)(z) = \bigvee_{z=x\vee y} \{f(x) \wedge g(y)\}, \qquad f \Subset g \text{ if } f \sqcup g = g,$$

$$\widetilde{a}(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for all } a \in [0, 1].$$

Two auxiliary unary operations were proposed on the algebra of fuzzy truth values to investigate extended minimum and extended maximum [19]. For all $f \in \mathscr{F}_2$ and $x \in [0, 1]$, two fuzzy truth values f^L and f^R are defined respectively as follows:

$$f^L(x) = \bigvee_{y\leqslant x} f(y) \text{ and } f^R(x) = \bigvee_{y\geqslant x} f(y).$$

A fuzzy truth value f is said to be *convex*, if for all $x \leqslant z \leqslant y$, $f(x) \wedge f(y) \leqslant f(z)$, or equivalently if $f = f^L \cap f^R$ [19]. A fuzzy truth value f is said to be *normal*, if $\bigvee_{x\in[0,1]} f(x) = 1$. The family of all convex (resp. normal, convex normal) fuzzy truth values is denoted as $\mathscr{F}_C$ (resp. $\mathscr{F}_N$, $\mathscr{F}_{CN}$). A fuzzy truth value f is said to be *upper semicontinuous* if α-cut set f_α is closed for all $\alpha \in [0, 1]$, where $f_\alpha = \{x \in [0, 1] | f(x) \geqslant \alpha\}$. The family of all convex normal and upper semicontinuous fuzzy truth values is denoted as $\mathscr{F}_{CNU}$. In fact, $f \in \mathscr{F}_{CNU}$ if and only if f_α is a closed interval for all $\alpha \in (0, 1]$, which is called a *fuzzy value* in [20].

Notice that two partial orders $\sqsubseteq$ and $\Subset$ are not equivalent to each other on the algebra of fuzzy truth values. Walker and Walker [19] pointed out that those two partial orders coincide with each other on the algebra of convex normal fuzzy truth values, which is a complete distributive lattice. In the sequel, we always apply $\sqsubseteq$ to denote the partial order on the (sub)algebra of convex normal fuzzy truth values.

Definition 2.1 [3,24]**.** Let $(P, \preceq, 0, 1)$ be a partially ordered set with a bottom element 0 and a top element 1. A mapping T: $P \times P \to P$ (resp. S: $P \times P \to P$) is called a *t-norm* (resp. *t-conorm*) on P if it is commutative, associative, increasing in each argument with respect to the partial order $\preceq$, and has the unit element 1 (resp. 0).

A t-norm T on a complete lattice P is left and right continuous if for all $x \in P$ and $\{x_i\}_{i\in I} \subseteq P$,

$$T\left(x, \bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} T(x, x_i) \text{ and } T\left(x, \bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} T(x, x_i), \text{ respectively,}$$

where I denotes a nonempty index set. Moreover, T is called continuous if it is simultaneously left and right continuous.

Obviously, $([0, 1], \leqslant)$ is a complete lattice with respect to the partial order $\leqslant$. Thus ordinary t-norms and t-conorms on [0, 1] can be viewed as the t-norms and t-conorms on the complete lattice $([0, 1], \leqslant)$.

Definition 2.2 [12]**.** A t-norm is said to be

(1) *Archimedean*, if for all $x, y \in (0, 1)$, there exists an $n \in \mathbf{N}$ such that $x_T^{(n)} < y$;
(2) *strict*, if it is continuous and strictly monotone;
(3) *nilpotent*, if it is continuous and there exists an $n \in \mathbf{N}$ such that $x_T^{(n)} = 0$;