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Some formal relationships among soft sets, fuzzy sets, and their extensions

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ABSTRACT

We prove that every hesitant fuzzy set on a set E can be considered either a soft set over the universe $[0, 1]$ or a soft set over the universe E. Concerning converse relationships, for denumerable universes we prove that any soft set can be considered even a fuzzy set. Relatedly, we demonstrate that every hesitant fuzzy soft set can be identified with a soft set, thus a formal coincidence of both notions is brought to light. Coupled with known relationships, our results prove that interval type-2 fuzzy sets and interval-valued fuzzy sets can be considered as soft sets over the universe $[0, 1]$. Altogether we contribute to a more complete understanding of the relationships among various theories that capture vagueness and imprecision.

1. Introduction

This paper investigates the formal relationships among notions arising from the theories of soft sets and fuzzy sets. In our analysis we include extended theories that introduce fuzziness and hesitancy (in the first instance) or only hesitancy (in the second one). Our paper complements previous successful contributions by other scholars, of which the following are a sample. Basu et al. [3] introduce soft set theoretic structures and show that they are equivalent to L-set theoretic structures. Bustince and Burillo [6] prove that vague sets are intuitionistic fuzzy sets. Yao [30] studies relationships and differences between theories of fuzzy sets and rough sets, with respect to two formulations of fuzzy sets and two views of rough sets. Kerre [16] contains a summary of links between fuzzy sets and other mathematical models (like flou sets, two-fold fuzzy sets and L-fuzzy sets). Torra [26] shows that all hesitant fuzzy sets can be represented as fuzzy multisets (Lemma 14) and as type-2 fuzzy sets (Lemma 16). Deschrijver and Kerre [9] establish relationships between intuitionistic fuzzy sets, L-fuzzy sets, interval-valued fuzzy sets, and interval-valued intuitionistic fuzzy sets. An overview of the mathematical relationships between intuitionistic fuzzy sets and other theories modeling imprecision is given by Deschrijver and Kerre [10]. Bustince et al. [5] provide a historical survey of types of fuzzy sets and their relationships. They also show that the original mathematical formulation of an interval type-2 fuzzy set (resp., Atanassov intuitionistic fuzzy set, vague set, grey set, interval-valued fuzzy set, interval-valued Atanassov intuitionistic fuzzy set) corresponds to a set-valued fuzzy set or a hesitant fuzzy set. In addition, set-valued fuzzy sets can be seen as type-2 fuzzy sets [5, Section V.A]. Bustince et al. [7] show that fuzzy sets and interval-valued fuzzy sets are particular cases of interval type-2 fuzzy sets.

Irrespective of their interactions, the purpose of all these extended theories is to capture subjectivity, uncertainty, imprecision of the appraisals, ... in order to better handle practical situations in applied fields. Zadeh [31] introduced fuzzy set

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theory, which meant a paradigm shift in mathematics by allowing partial membership. There is a vast literature on fuzzy sets and their applications, including a number of successful generalizations. In particular, Torra [26] extended this theory by introducing hesitant fuzzy sets. Bustince et al. [5] explain that the notion of hesitant fuzzy set coincides with the notion of set-valued fuzzy set in Grattan–Guinness [15]. Their usefulness derives from the necessity of modeling imprecise human knowledge (like collective knowledge) which cannot be suitably represented by fuzzy sets. A good account of existing results and applications of hesitant fuzzy sets was recently done by Rodríguez et al. [25]. From a different perspective, Molodtsov [24] initiated the theory of soft sets. He showed its applicability to several fields and established some fundamental results. Further developments appear in Maji et al. [23], Aktaş and Çağman [1], Maji, Biswas and Roy [21] (who introduce fuzzy soft sets), Wang, Li and Chen [27] (who introduce hesitant fuzzy soft sets), or Li and Xie [18] (who prove that there is a one-to-one correspondence between the set of all fuzzy soft sets and the set of all $[0, 1]$-valued information systems).

Interestingly, Molodtsov [24] showed that the models by fuzzy sets and soft sets are not independent. He proved that every fuzzy set can be considered a soft set (see also Maji et al. [22]). Here we extend this conclusion and show that even every hesitant fuzzy set can be considered a soft set over the common universe $[0, 1]$. Alternatively, by the recourse to α-level cuts we prove that every hesitant fuzzy set on E can be considered a soft set $(\phi, [0, 1])$ over the universe E. Concerning converse relationships, we prove that every soft set on a denumerable universe – i.e., either finite or of the cardinality of the set of natural numbers – can be considered a fuzzy set on its set of parameters. Relatedly, we prove that every hesitant fuzzy soft set can be identified with a soft set, thus a formal coincidence of both notions is brought to light. Some particularizations of the aforementioned notions give raise to other specific identifications.

When we couple these results with other known relationships we obtain new insights. For example, that every interval type-2 fuzzy set, resp., interval-valued fuzzy set, can be identified with a soft set over $[0, 1]$.

Joining all these results, we obtain a rich picture of the relationships among various theories that capture vagueness and imprecision, which fills gaps in recent surveys like the aforementioned [5] and [25].

This paper is organized as follows. Section 2 recalls some notation and definitions like fuzzy sets, hesitant fuzzy sets, soft sets, fuzzy soft sets, and hesitant fuzzy soft sets. We also brief the reader on some basic facts about Cartesian products. In Section 3 we present our main results, including examples that illustrate the identifications proved. In Section 4 we check for other implications of our results in view of relationships proven by recent literature. They concern the notions of interval type-2 fuzzy set and interval-valued fuzzy set. We conclude in Section 5.

2. Notation and definitions

In this section we first fix some notations and recall definitions in (crisp) set theory. Then we recall concepts from fuzzy sets, soft sets, and some of their extensions.

2.1. Crisp sets, subsets, and products of sets

For any set U, $\mathcal{P}^*(U)$ denotes the set of non-empty subsets of U, $\mathcal{P}(U)$ denotes the set of all subsets of U, and $\mathcal{F}^*(U)$ denotes the set of non-empty finite subsets of U.

We make extensive use of the formal concept of Cartesian product of an indexed family of sets. Our exposition closely follows Willard [28, paragraphs 8.1 and 8.2 c)]. Suppose a non-empty family $\{X_o\}_{o\in U}$ of non-empty sets, that is, each X_o is a non-empty set and the family is indexed by the non-empty set U. The Cartesian product of the family is the set

$$\prod_{o\in U} X_o = \left\{ x : U \longrightarrow \bigcup_{o\in U} X_o \text{ such that } x(o) \in X_o \text{ for each } o \in U \right\}.$$

When $X_o = X$ for each $o \in U$, then $\prod_{o\in U} X_o = \prod_{o\in U} X$ is just the set of all functions $x : U \longrightarrow X$, which is commonly denoted as X^U.

Remark 1. Willard [28, paragraph 1.5 and Problem 1.D] explains how this formal concept adapts to the finite case and results into our familiar notion of a Cartesian product as a set of ordered finite-length vectors. Informally, one works with the idea that the elements of $\prod_{o\in U} X_o$ are U-ordered tuples $(x_o)_{o\in U}$ where each x_o belongs to X_o. To this purpose one simply identifies each $x : U \longrightarrow \bigcup_{o\in U} X_o$ with $(x(o))_{o\in U}$.

2.2. Fuzzy and hesitant fuzzy sets

Henceforth $\mathbf{FS}(X)$ denotes Zadeh's fuzzy subsets of X. Recall that a fuzzy subset (FS) A of X is characterized by a function $\mu_A : X \to [0, 1]$. When $x \in X$, the number $\mu_A(x) \in [0, 1]$ is called the degree of membership of x in the subset. It represents the degree of truth of the statement "x belongs to A".

The concept of a hesitant fuzzy set leans on the previous concept of hesitant fuzzy elements:

Definition 1. (See Xia and Xu [29].) A hesitant fuzzy element (HFE) is a non-empty, finite subset of $[0, 1]$. The set of HFEs is denoted by $\mathcal{F}^*([0, 1])$.