Journal of Symbolic Computation 75 (2016) 94–109

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Logspace computations in graph products

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A R T I C L E I N F O

Article history:
Received 6 December 2014
Accepted 10 July 2015
Available online 4 November 2015

Keywords:
Algorithmic group theory
Graph products
Logspace complexity
Word problem
Geodesics
Conjugacy problem

A B S T R A C T

We consider three important and well-studied algorithmic problems in group theory: the word, geodesic, and conjugacy problem. We show transfer results from individual groups to graph products. We concentrate on logspace complexity because the challenge is actually in small complexity classes, only. The most difficult transfer result is for the conjugacy problem. We have a general result for graph products, but even in the special case of a graph group the result is new. Graph groups are closely linked to the theory of Mazurkiewicz traces which form an algebraic model for concurrent processes. Our proofs are combinatorial and based on well-known concepts in trace theory. We also use rewriting techniques over traces. For the group-theoretical part we apply Bass–Serre theory. But as we need explicit formulae and as we design concrete algorithms all our group-theoretical calculations are completely explicit and accessible to non-specialists.

1. Introduction

Background. Algorithmic questions concerning finitely generated groups have been studied for more than 100 years starting with the fundamental work of Tietze and Dehn in the beginning of the 20th century. In this paper we investigate three algorithmic problems for graph products G with a finite and symmetric generating set Σ. The question for us is whether they can be decided in logspace.

(1) **Word problem.** Let $w \in \Sigma^*$. Is $w = 1$ in the group G?

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http://dx.doi.org/10.1016/j.jsc.2015.11.009

(2) **Geodesic problem.** Let $w \in \Sigma^*$. Compute a geodesic, i.e., a shortest word representing $w \in G$; and, if a linear order on Σ is defined, compute the lexicographical first word among all geodesics, i.e., compute a shortlex normal form of w.
(3) **Conjugacy problem.** Let $u, v \in \Sigma^*$. Are u and v conjugated in G?

The complexity of the first and third problem depends on G only, whereas for the second problem we have to specify $\Sigma \subseteq G$, too. Over the past few decades the search and design of algorithms for decision problems like the ones above has developed into an active research area, where algebraic methods and computer science techniques join in a fruitful way, see for example the recent surveys (Lohrey, 2012; Sapir, 2011). Of particular interest are those problems which can be solved efficiently in parallel. More precisely, we are interested in *deterministic logspace*, called simply *logspace* in the following. This is a complexity class at the lower level in the NC-hierarchy[1]:

$$\mathsf{NC}^1 \subseteq \mathsf{logspace} \subseteq \mathsf{LOGCFL} \subseteq \mathsf{NC}^2 \subseteq \cdots \subseteq \mathsf{NC} = \bigcup_{i \geq 1} \mathsf{NC}^i \subseteq \mathsf{P} \subseteq \mathsf{NP}. \tag{1}$$

No separation result between NC^1 and NP is known but it is believed (by some) that all of the above inclusions in (1) are strict. A fundamental result in the context of group-theoretical algorithms was shown by Lipton and Zalcstein (1977) and Simon (1979): the word problem of finitely generated linear groups belongs to logspace. As a special case, the word problem of the group F_2 of rank 2 is in logspace. The class of groups with a word problem in logspace is further investigated by Waack (1981). Another important result due to Cai (1992) (resp. Lohrey (2005)) is that the word problem of hyperbolic groups is in NC^2 (resp. in LOGCFL). The class LOGCFL coincides with the (uniform) class SAC^1. It is a subclass of NC^2. Often, it is not enough to solve the word problem, but one has to compute a normal form. This leads to the problem of computing geodesics. This problem and various related problems were studied e.g. in (Droms et al., 1993; Elder, 2010; Elder and Rechnitzer, 2010; Myasnikov et al., 2010; Paterson and Razborov, 1991). These results imply that there are groups with an easy word problem (in logspace), but where simple questions related to geodesics are computationally hard, for example NP-complete for certain wreath products or free metabelian groups of rank 2. Droms et al. (1993) developed the techniques for the geodesic problem in free solvable groups, but Droms et al. (1993) wrongly stated that one can find geodesics in polynomial time. The correct version of the result, namely that finding geodesics is NP-complete was shown by Myasnikov et al. (2010). Myasnikov et al. (2010) use the techniques of Droms et al. (1993) and reduced the problem of finding Steiner trees in a grid to finding geodesics in a certain free solvable group. Finally, the conjugacy problem is a classical decision problem which is notoriously more difficult than the word problem. Whereas for a wide range of groups the word problem is decidable (and often easily decidable) the conjugacy problem is not known to be decidable. This includes e.g. automatic groups (word problem is in $\mathcal{O}(n^2)$) or one-relator groups (word problem is decidable) to mention two classes. Miller's group (Miller III, 1971) has a decidable word problem (at most cubic time,[2] actually logspace) but undecidable conjugacy problem. Actually, there are finitely generated subgroups of $F_2 \times F_2$ (hence subgroups of $\mathrm{SL}(4, \mathbb{Z})$, hence linear groups) with unsolvable conjugacy problem (Miller III, 1992, Thm. 5.2).

Here, we continue and generalize the work of Diekert et al. (2012) from graph groups to graph products of groups having a word problem in logspace. We show transfer results for all three problems mentioned above. However, techniques of Diekert et al. (2012) for graph groups (which used linear representations for right-angled Coxeter groups) are not available in the present paper, simply because linear representations do not exist for the individual groups, in general. For graph products we start with a list L of groups G_α. Next, we endow $\mathcal{L}$ with an irreflexive and symmetric relation $I \subseteq \mathcal{L} \times \mathcal{L}$. This means $(\mathcal{L}, I)$ is a finite undirected graph and each node $\alpha \in \mathcal{L}$ is associated with a node group G_α. The graph product G is then the free product of the G_α's modulo defining

[1] NC^i is the class of languages which are accepted by (uniform) boolean circuits of polynomial size, depth $O(\log^i(n))$ and constant fan-in, see e.g. (Vollmer, 1999) for a textbook.

[2] Mark Sapir, personal communication.