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Fuzzy relations between Dempster–Shafer belief structures

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ABSTRACT

Relationships between attribute values can involve concepts such as similar or greater then. We first discuss the mathematical structure of relationships. We consider both crisp and fuzzy relationships. We introduce some important examples of mathematical relationships and describe their formal properties. We then turn to the problem of calculating the degree of relationship between attributes whose values are uncertain. We first consider the case when the uncertainties in the attribute values are expressed in terms of possibility distributions. We next look at the case when the uncertainties are in the form of a standard Dempster–Shafer belief structure. We finally consider the situation when the uncertain information associated with the attribute values are expressed in terms of Dempster-Shafer belief structures whose focal elements are fuzzy sets.

1. Introduction

Relationships such as “greater then” and “about equal” are pervasive in human reasoning and decision-making and have a long history in mathematics [1–3]. Beginning with his pioneering paper [4] Zadeh has shown the important role that fuzzy mathematics can play modeling the kinds of soft relationship commonplace in human cognition. Other researchers [5–10] have significantly followed the path initiated by Zadeh. If we have a relationship R, for example, corresponding to “approximately the same” with regard to concept age then if A and B are two people whose ages we know we can easily calculate the degree to which A and B’s are approximately the same. Our concern here is with a slightly more complex problem. Here rather than knowing the exact values of the two people’s ages our knowledge of their ages is uncertain. Thus here we have two uncertain variables U and V corresponding to A and B’s ages and we are interested in determining the degree to which these uncertain variables satisfy the relationship of being an approximately the same. Here we look at this problem for cases where the uncertain information takes the form of a possibility distribution as well as a Dempster–Shafer belief structure [11–17].

2. Fuzzy relationships

A relationship is a fundamental mathematical concept based on the idea of the Cartesian product. The Cartesian product of two sets X and Y, $X \times Y$, is the set of all pairs containing one element from each of the sets. Thus if $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ then their Cartesian product $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$. A relationship between elements in X and Y is a subset $\boldsymbol{R} \subseteq X \times Y$. In the preceding an example of a relationship would be the set $\{(x_1, y_1), (x_1, y_2), (x_2, y_2)\}$. A relationship can be characterized by a function R: $X \times Y \rightarrow \{0, 1\}$ such that $R(x, y) = 1$ if $(x, y) \in \boldsymbol{R}$ and $R(x, y) = 0$ if $(x, y) \notin \mathbf{R}$.

A fuzzy relationship between the sets X and Y is a mapping R: $X \times Y \rightarrow [0, 1]$. [4, 18, 19] It associates with each pair of points in the Cartesian product $X \times Y$ a number in the unit interval. Formally a fuzzy relationship can be seen as fuzzy subset of the Cartesian set $X \times Y$. Here $R(x, y)$ is the membership grade of the element (x, y) in the fuzzy set **R**. Semantically $R(x, y)$ can be seen as the degree of **R**-relationship between x and y.

A fuzzy relationship on two finite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ can be viewed as a matrix:

$$R = \begin{bmatrix} R(x_1, y_1) & \ldots\ldots. & R(x_1, y_m) \\ R(x_2, y_1) & \ldots\ldots. & R(x_2, y_m) \\ . & \ldots\ldots & . \\ . & \ldots\ldots & . \\ R(x_n, y_1) & \ldots\ldots. & R(x_n, y_m) \end{bmatrix}$$

Fuzzy relationships have been used in many domains to represent various types of information. For example assume X is a set of people and Y is a collection of baseball teams. We can have a relationship **R** such that $R(x, y)$ is the degree to which x is a fan of

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team y. A matrix view of such a relationship would be

$$\begin{array}{c} \\ \text{John} \\ \text{Bill} \\ \text{Tom} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Yanks} & \text{Mets} & \text{Reds} & \text{A's} \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0 & 0.3 \\ 0 & 1 & 1 & 0 \\ 0.7 & 0.3 & 0.6 & 0.9 \end{bmatrix} \end{array}$$

We see in such a relationship there is no specific property that the relationship R needs to satisfy other than that the values R(x, y) lie in the unit interval, it is arbitrary.

Social networks can be modeled using fuzzy relationships [20–24]. We see if X is a collection of nodes then here we can have a relationship R on $X \times X$ such that for each x and $y \in X$ the value R(x, y) could be a formulation of the degree of friendship between node x and y. Here again the relationship R can be arbitrary. Here we are just representing the state of the world.

Mathematical relationships such as "much greater than" can be modeled as fuzzy relationships. Let $X = \{1, 2, 3, 4\}$ and consider the relationship "much greater than". Here we can model this with a fuzzy relationship on $X \times X$ shown below where R(x, y) indicates the degree to which x is "much greater than" y:

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 1 & 0.6 & 0.4 & 0 \end{bmatrix} \end{array}$$

This relationship is not completely arbitrary as it has the special requirement that if $x \le y$ then $R(x, y) = 0$.

The relationship "approximately equal" could be expressed using the relationship

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & .8 & 0.2 & 0 \\ 0.8 & 1 & 0.7 & 0.2 \\ 0.2 & 0.7 & 1 & 0.9 \\ 0 & 0.2 & 0.9 & 1 \end{bmatrix} \end{array}$$

Here R(x, y) indicates the degree to which x is approximately equal to y. This relationship has some special properties. Among these is $R(x, x) = 1$ and symmetry $R(x, y) = R(y, x)$.

Another type of relationship, one that is useful in decision-making, is a preference relationship. Assume $X = \{x_1, x_2, x_3, x_4\}$ is a collection of alternatives then we can use a relationship R on $X \times X$ so that R(x, y) indicates the degree to which x is preferred to y.

Since fuzzy relationships are fuzzy sets it is possible to perform all kinds of fuzzy set operation on them [18]. Thus if R_1 and R_2 are two fuzzy relationships on $X \times Y$ we can obtain the union and intersection of these relationships. Here then if $R_3 = R_1 \cap R_2$ then $R_3(x, y) = R_1(x, y) \wedge R_2(x, y)$ and if $R_4 = R_1 \cup R_2$ then $R_4(x, y) = R_1(x, y) \vee R_2(x, y)$. We can obtain the negation of the relationship R_1 such that not $(R1) = \bar{R}_1$ where $\bar{R}_1(x, y) = 1 - R1(x, y)$. We can also define containment. Here $R_1 \subseteq R_2$ if $R_1 = (x, y) \le R_2(x, y)$ for all pairs of (x, y). If we denote $R^* = X \times Y$ then all fuzzy relationships R on $X \times Y$ are such that $R \subseteq R^*$. If R_1 and R_2 are two fuzzy relationships on $X \times Y$ such that $R_1 = \subseteq R_2$ then we shall say R_2 is a bigger relationship.

While all fuzzy set operations are available to relationships there are some operations that are unique to relationships. One of these is inverse or transform, if R is a relationship then R^{-1} is a relationship such that $R^{-1}(x, y) = R(y, x)$.

Another operation special to relationships is composition. Assume R is a relationship on $X \times Y$ and Q is a relationship on $Y \times Z$. The max-min composition of R and Q denoted $R \bullet Q$ is a fuzzy relationship S defined on $X \times Z$ such that $S(x, z) = \text{Max}_y [R(x, y) \wedge Q(y, z)]$. Other forms of composition are available such as max-product composition $S(x, z) = \text{Max}_y [R(x, y) \cdot Q(y, z)]$. Actually if t is a t-norm we can define Max-t composition so that $S(x, z) = \text{Max}_y[t(R(x, y), Q(y, z))]$ [25].

Another operation special to relationships is projection. This allows us to obtain a fuzzy subset over a constituent space. Assume R is a relationship on $X \times Y$. The projection of R onto X, denoted $\text{Proj}(R \downarrow X)$, induces, a fuzzy subset F on X so that $F(x) = \text{Max}_{y \in Y} [R(x, y)]$. Here then we see that F(x) is the maximal membership grade of any element in R containing x. Similarly the projection of R onto Y, denoted $\text{Proj}(R \downarrow Y)$, is defined as a fuzzy subset E on Y so that $E(y) = \text{Max}_{x \in X}[R(x, y)]$.

We shall say a relationship R on $X \times Y$ is normal in X if for each $x \in X$ there exists at least one $y \in Y$ such that $R(x, y) = 1$. Similarly we can define a relationship as being normal in Y. We easily see that if R is normal in X then $\text{Proj}(R \downarrow X)$ is X. We also note that if R is normal in Y then its projection onto Y is Y, $\text{Proj}(R \downarrow Y) = Y$.

Assume A and B are fuzzy subsets respectively on the spaces X and Y their Cartesian product, $A \times B$, is a relationship R on $X \times Y$ such that $R(x, y) = \text{Min}(A(x), B(y)) = A(x) \wedge B(y)$. A special case of Cartesian product is the cylindrical extension of a fuzzy set [26, 27]. Assume A is a fuzzy subset on X its cylindrical extension with respect to Y, denoted $\text{Cyl}(A \uparrow Y)$, is a relationship R on $X \times Y$ so that $R(x, y) = A(x) \wedge Y(y)$. Here we see that $R(x, y) = A(x)$ for all x. Similarly we could define the cylindrical extension of a fuzzy set E of Y.

It is interesting to note that if R is a relationship on $X \times Y$ and $A = \text{Proj}(R \downarrow X)$ and if R_1 is the cylindrical extension of A onto $X \times Y$ then $R \subseteq R_1$. That is R_1 is a bigger relationship.

We can consider non-standard fuzzy relationships in which the membership grades of R(x, y) can be interval valued, intuitionistic or Pythagorean membership grades [28–32]. However, unless otherwise specified we shall assume the standard case where the membership grades are drawn from the unit interval.

A very special case of fuzzy relationship R occurs when $X = Y$. Here we refer to R as a fuzzy binary relationship on X. In [4], Zadeh studied these relationships in considerable detail.

A considerable amount of literature exists on describing features for these types of binary relationships in the crisp environment [2, 33]. Many of these features have been extended to the fuzzy environment. In the following we describe some of these features associated with fuzzy binary relationships [4]. In the following we assume R in a fuzzy binary relationship on X.

(1) R is called **reflexive** if $R(x, x) = 1$ for all $x \in X$
(2) R is called **irreflexive** $R(x, x) \neq 1$ for some $x \in X$
(3) R is called **anti-reflexive** if $R(x, x) \neq 1$ for all $x \in X$
(4) R is called **symmetrical** if $R(x, y) = R(y, x)$ for all pairs x and y
(5) R is called **asymmetrical** if $R(x, y) \neq R(y, x)$ for some pairs x and y
(6) R is called **anti-symmetric** if $R(x, y) > 0$ and $R(y, x) > 0$ implies $x = y$
(7) R is called **transitive** if for all pairs $(x, z) \in X \times X$

$$R(x, z) \geq \text{Max}_{y \in X} [R(x, y) \wedge R(y, z)]$$

(8) If $R(x, z) < \text{Max}_{y \in X} [R(x, y) \wedge R(y, z)]$ for some pair x, z then R is called **non-transitive**
(9) R is called **anti-transitive** if for all pairs $(x, z), R(x, z) < \text{Max}_{y \in X} [R(x, y) \wedge R(y, z)$

Various special relationships can be defined by combining a number of these features. A fuzzy relationship R on X is called a **similarity relationship** if it is: 1) reflexive 2) symmetric and 3) transitive. We note that a similarity relation extends the binary concept of equality. The following relation R_S provided by Zadeh in [4] is an example of a similarity relation. Here for each pair (x_i, x_j) the value of $R_S(x_i, x_j)$ indicates the degree to x_i and x_j are sim-