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On concept reduction based on some graph properties

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ABSTRACT

The theory of concept lattices represents a well established and widely used conceptual data-mining method. Considering additional information represented by a graph structure on a set of objects, we propose a reduction of concepts. Using graph-theoretical point of view on FCA together with simple probabilistic arguments we derive the mean value of the cardinality of the reduced hierarchical structure.

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1. Introduction

Formal concept analysis (FCA), with its roots in the algebraic theory of ordered sets, provides a mathematical framework for data analysis and knowledge discovery. As an effective tool, FCA has been applied to a variety of fields such as decision making, data mining, information retrieval or machine learning.

The theory of FCA is based on the notions of formal context and formal concept respectively. A formal context is a triple (O, A, I) where O, A are non-empty sets and $I \subseteq O \times A$ is a binary relation. The formal context (O, A, I) formalizes the notion of object-attribute model, where O is interpreted as the set of objects, A the set of attributes (properties), and $(o, a) \in I$ is interpreted as the object o has the attribute (property) a. A concept can be formally seen as an extent-intent pair. The extent is understood as the collection of all objects belonging to the concept, while the intent is the multitude of all attributes common to all those objects. The set of all formal concepts of a formal context forms a complete lattice, called the concept lattice, which reflects the relationship of generalization and specialization among the formal concepts.

The notion of a formal context is essentially very simple and it also admits a graph-theoretic interpretation. A graph, among all representation forms, is a way to present data or knowledge. One of the major advantages of graph-based models is the presence of structural information embedded in the graphs. Assuming that an additional graph structure is given on a set of objects, one can consider only such concepts where extents satisfy a prescribed graph property. This can lead to a significant reduction of the concept lattice. Let us mention that the problem of the concept lattice reduction, or some kind of knowledge reduction, plays an important role in the recent research of the classical concept lattices ([7,20]) as well as fuzzy concept lattices cf. [1–4,10,11,18,19,21,22,24–28].

A natural idea to constrain a concept lattice by a closure system appeared e.g. in [4]. We use the same idea, however for restricted closure systems given by independent systems induced by graph structure on objects. Our main aim is to study a reduction effect from the statistical point of view, i.e., to quantitatively describe a reduction effect in an average case. We provide an exact characteristic of the mean value of number of concepts in randomly graph based formal contexts. Initially, in order to describe a general situation, we will consider the so-called independent systems and concept lattices constrained by such systems. The presented construction involves a composition of a closure operator with the classical concept forming operators induced by a formal context. This leads to the so-called weak Galois connections which provide more general framework for defining concept-lattice-like structures than the Galois connections do. Also in this case, the concepts represent combinations of objects

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and attributes that are ôclosedö in a certain sense. After introducing the general theory, we focus on two independent systems induced by a graph structure on the set of all objects. In particular we will consider clique independent system determined by maximal cliques and connected independent system determined by connected components.

2. Preliminaries

In this paper all considered sets are assumed to be finite. The theory of concept lattices is based on the notion of Galois connection, hence we recall this notion first, cf. [13]. Let $(P, \leq)$ and $(Q, \leq)$ be the ordered sets and let

$$\varphi : P \to Q \quad \text{and} \quad \psi : Q \to P$$

be maps between these ordered sets. Such a pair (φ, ψ) of mappings is called a Galois connection if:

(a) $p_1 \leq p_2 \Longrightarrow \varphi(p_1) \geq \varphi(p_2)$,
(b) $q_1 \leq q_2 \Longrightarrow \psi(q_1) \geq \psi(q_2)$,
(c) $p \leq \psi(\varphi(p))$ and $q \leq \varphi(\psi(q))$.

We note that any Galois connection (φ, ψ) between P and Q satisfies

$$\varphi(p) = \varphi(\psi(\varphi(p))) \quad \text{and} \quad \psi(q) = \psi(\varphi(\psi(q)))$$

for each $p \in P$ and $q \in Q$.

Galois connections are closely related to the notions of closure operator and closure system. A closure operator on a set S is a function c: $\mathbf{P}(S) \to \mathbf{P}(S)$ from the power set of S to itself which is extensive, increasing and idempotent, i.e., it satisfies the following conditions for all subsets $X, Y \subseteq S$:

(1) $X \subseteq c(X)$,
(2) $X \subseteq Y \Longrightarrow c(X) \subseteq c(Y)$,
(3) $c(c(X)) = c(X)$.

A family of subsets $\mathcal{F} \subseteq \mathbf{P}(S)$is called a closure system, if $\mathcal{F}$satisfies the following two properties:

(1) $S \in \mathcal{F}$,
(2) $X, Y \in \mathcal{F} \implies X \cap Y \in \mathcal{F}$.

The sets of a closure system $\mathcal{F}$are called closed sets of $\mathcal{F}$. A closure system $\mathcal{F}$forms a lattice with respect to set inclusion. In this case, lattice operations $\wedge$ and $\vee$ are given by

$$X \wedge Y = X \cap Y \quad X \vee Y = \bigcap\{Z \in \mathcal{F} : X \cup Y \subseteq Z\}.$$

Let us remark that closure operators are in one-to-one correspondence with closure systems by the map $c \mapsto \mathrm{FP}(c) = \{X \subseteq S : c(X) = X\}$which associates to a closure operator c the family of all its fixed points which forms a closure system. Conversely, the inverse map associates to each closure system $\mathcal{F}$the closure operator $c_{\mathcal{F}}$defined by $c_{\mathcal{F}}(X) = \bigcap\{Y \in \mathcal{F} : X \subseteq Y\}$.

Let A, B be sets and a pair φ, ψ be a Galois connection between the power sets $\mathbf{P}(A)$ and $\mathbf{P}(B)$. It is well known that the composition of φ and ψ, $\psi \circ \varphi$: $\mathbf{P}(A) \to \mathbf{P}(A)$ forms a closure operator on the set A. Similarly, $\varphi \circ \psi$: $\mathbf{P}(B) \to \mathbf{P}(B)$ forms a closure operator on the set B. Moreover, the corresponding closure systems are dually isomorphic as posets.

Next, we briefly describe the basics about FCA. The notion of formal context provides a framework for this theory. A formal context is a triple (O, A, I) where O, A are nonempty sets, $I \subseteq O \times A$ is a relation from O to A. In a formal context (O, A, I), O is interpreted as the set of objects, A the set of attributes (properties), and $(o, a) \in I$ reads as that the object o has the property a.

Let (O, A, I) be a formal context. There is a pair of mappings $^{\uparrow}$: $\mathbf{P}(O) \to \mathbf{P}(A)$ and $^{\downarrow}$: $\mathbf{P}(A) \to \mathbf{P}(O)$, which forms a Galois connection between power sets of O and A, respectively.

$$X^{\uparrow} = \{a \in A : (o, a) \in I, \forall o \in X\},$$

$$Y^{\downarrow} = \{o \in O : (o, a) \in I, \forall a \in Y\}.$$

A formal concept of the formal context (O, A, I) is a pair (X, Y) with $X \subseteq G$, $Y \subseteq M$, $X^{\uparrow} = Y$and $X = Y^{\downarrow}$. The set $X$is called the extent and Y is called the intent of the concept (X, Y). If (X_1, Y_1) and (X_2, Y_2) are concepts, then (X_1, Y_1) is a subconcept of (X_2, Y_2), if $X_1 \subseteq X_2$ (which is equivalent to $Y_1 \supseteq Y_2$). This fact is denoted by $(X_1, Y_1) \leq (X_2, Y_2)$. The relation $\leq$ is called the hierarchical order of concepts and the structure of all concepts ordered by this relation is denoted by $\underline{\mathfrak{B}}(O, A, I)$.

Let (O, A, I) be a formal context. Any extent X of the concept (X, Y) satisfies $X = Y^{\downarrow} = X^{\uparrow\downarrow}$,hence it is closed under the composition of operators $^{\uparrow}$ and $^{\downarrow}$. Conversely, if X is closed, then the pair $(X, X^{\uparrow})$ forms a concept. Hence the set of all extents of $\underline{\mathfrak{B}}(O, A, I)$forms a closure system on the set O of all objects. The order structure of $\underline{\mathfrak{B}}(O, A, I)$is uniquely determined by the order structure of extents, thus $\underline{\mathfrak{B}}(O, A, I)$forms a lattice, referred as the concept lattice. In this case, the lattice operations $\wedge$ and $\vee$ are given by

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, (X_1 \cap X_2)^{\uparrow}) = (X_1 \cap X_2, (Y_1 \cup Y_2)^{\downarrow\uparrow})$$

$$(X_1, Y_1) \vee (X_2, Y_2) = ((Y_1 \cap Y_2)^{\downarrow}, Y_1 \cap Y_2) = ((X_1 \cup X_2)^{\uparrow\downarrow}, Y_1 \cap Y_2).$$

An analogy of concept lattices can be defined in more general framework than provide Galois connections. One of such possibility is described in [23], where the so-called weak Galois connections are presented.

Let P and Q be partially ordered sets. A pair of mappings α: $P \to Q$, β: $Q \to P$ is said to be a weak Galois connection between P and Q if it satisfies:

(i) $x_1 \leq x_2$ implies $\alpha(x_1) \geq \alpha(x_2)$,
(ii) $y_1 \leq y_2$ implies $\beta(y_1) \geq \beta(y_2)$,
(iii) $\beta(\alpha(x)) = \beta(\alpha(\beta(\alpha(x))))$and $\alpha(\beta(y)) = \alpha(\beta(\alpha(\beta(y))))$.

Similarly, as in the case of Galois connections, if α and β is a weak Galois connection between $\mathbf{P}(O)$ and $\mathbf{P}(A)$, a concept lattices like structure can be defined. Denote by $\mathcal{B}(O, A, \alpha, \beta)$a subset of $\mathbf{P}(O) \times \mathbf{P}(A)$ such that

$$\mathcal{B}(O, A, \alpha, \beta) = \{(X, Y) : Y = \alpha(X) \text{ and } X = \beta(Y)\}.$$

A partial order $\leq$ on $\mathcal{B}(O, A, \alpha, \beta)$is defined in the same way as in the case of concept lattices, i.e.,

$$(X_1, Y_1) \leq (X_2, Y_2) \text{ iff } X_1 \subseteq X_2 \text{ iff } Y_1 \supseteq Y_2.$$

The following proposition shows that such defined structure has almost the same properties as concept lattices defined via Galois connections.

Proposition 2.1. *The structure $\mathcal{B}(O, A, \alpha, \beta)$forms a lattice, where the operation $\wedge$ and $\vee$ are given by*

$$(X_1, Y_1) \wedge (X_2, Y_2) = (\beta(\alpha(X_1 \cap X_2)), \alpha(\beta(Y_1 \cup Y_2)))$$

$$(X_1, Y_1) \vee (X_2, Y_2) = (\beta(\alpha(X_1 \cup X_2)), \alpha(\beta(Y_1 \cap Y_2))).$$

Proof. We prove the formula for infimum. Let $(X_1, Y_1), (X_2, Y_2) \in \mathcal{B}(O, A, \alpha, \beta)$be arbitrary elements. In order to simplify notations, denote $X = \beta(\alpha(X_1 \cap X_2))$and $Y = \alpha(\beta(Y_1 \cup Y_2))$. First, we show that $\alpha(X) = Y$and $\beta(Y) = X$,i.e.,

$$(\beta(\alpha(X_1 \cap X_2)), \alpha(\beta(Y_1 \cup Y_2))) \in \mathcal{B}(O, A, \alpha, \beta).$$

Since both mappings α and β are antitone, it follows that the compositions $\beta \circ \alpha$: $\mathbf{P}(O) \to \mathbf{P}(O)$ as well as $\alpha \circ \beta$: $\mathbf{P}(A) \to \mathbf{P}(A)$ are monotone. Consequently, $X = \beta(\alpha(X_1 \cap X_2)) \subseteq \beta(\alpha(X_1)) = \beta(Y_1) = X_1$and

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