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Some issues on the OWA aggregation with importance weighted arguments

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ABSTRACT

We introduce the OWA operator and note that it provides a parameterized class of aggregation operators. Here the parameterization is accomplished by the choice of the characterizing OWA weights, different characterizing weights results in different aggregation imperatives. We discuss various ways of providing these characterizing OWA weights. Most notable among these are the use of a vector containing the prescribed weights and the use of a function called the weight generating function from which the characterizing can be extracted. In many applications we are faced with situations in which the arguments being aggregated have different importances. This raises the issue of appropriately combining the individual argument weights with the characterizing weights of the operator to obtain operational weights to be used in the actual aggregation. Our goal here is looking at this issue under different methods of specification of the characterizing weights.

1. Introduction

In [1] we introduced the Ordered Weighted Average (OWA) operator. The OWA operator provides a parameterized class of mean-like operators which can be used to aggregate a collection of arguments. The parameterization is accomplished by the choice of the characterizing OWA weights that are multiplied by the argument values in a linear type aggregation. A unique feature of the OWA operator is that the association of these weights with the arguments is based on an ordering of the arguments determined by the argument magnitudes. One popular use of the OWA operator is for multi-criteria decision making, here the arguments are the satisfactions to each of the relevant criteria by a given decision alternative, *x*, and the aggregated value is the overall satisfaction of the alternative x to the collection of criteria. An important feature of a multi-criteria decision problem is the decision imperative, the procedure used for combining an alternative's satisfactions to the individual criteria to obtain its overall satisfaction to the decision problem. The parameterization of the OWA operator provides it with the ability to model in a unified manner various different types of decision imperatives by the choice of the characterizing OWA weights. Another aspect of the multi-criteria decision problem is that often the criteria have different importance weights. This raises the issue of how to combine these criteria importance weights with the characterizing OWA weights used to implement the decision imperative in a manner to that can be used in the OWA aggregation framework. Our interest is to look at this problem of OWA aggregation in the case of importance weighted arguments [1–9].

2. The OWA operator

In the following we provide the definition of the OWA aggregation operator.

Definition. An OWA operator of dimension n is defined in terms of a collection of n weights $\mathrm{w_j}$ for $j = 1$ to n such that each $w_j = [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. The OWA aggregation of a collection of numeric arguments, $(a_1, \ldots, a_n)$ is defined as $\mathrm{OWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j a_{id(j)}$ where *id* is an index function so that *id*(*j*) is the *j*th largest argument value.

The OWA operator can be shown to have the following properties [1]

(1) **Monotonicity**: $\mathrm{OWA}(a_1, \ldots, a_n) \geq \mathrm{OWA}(b_1, \ldots, b_n)$ if $a_i \geq b_i$ for all i
(2) **Boundedness:** $\mathrm{Min}_i[a_i] \leq \mathrm{OWA}(a_1, \ldots, a_n) \leq \mathrm{Max}_i[a_i]$
(3) **Symmetry**: The initial indexing of the arguments is irrelevant
(4) **Idempotency**: $\mathrm{OWA}(a_1, \ldots, a_n) = a$ if all $a_i = a$

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The satisfaction of these properties implies that the OWA operator is a mean or averaging operator [10].

In the special case when the arguments, the a_i, are contained in the interval [0, 1] then $\text{OWA}[a_1, \ldots, a_n] \in [0, 1]$ and then since in this case the OWA also has the following properties

(1) $\text{OWA}(0, \ldots, 0) = 0$
(2) $\text{OWA}(1, \ldots, 1) = 1$
(3) Monotonicity

then the OWA operator is also an aggregation operators as defined in [10–12].

We can provide a vector formulation of this operator. Let $W = \begin{bmatrix} w_1 \\ w_n \end{bmatrix}$ be an n-vector called the weighting vector. Let $B = \begin{bmatrix} a_{id(1)} \\ a_{id(n)} \end{bmatrix}$ be an n-vector called the ordered argument vector. Using these we can express $\text{OWA}(a, \ldots, a) = W^T B$

Example. Assume $n = 4$ and $w_1 = 0.3$, $w_2 = 0.2$, $w_3 = 0.4$, $w_4 = 0.1$ and our arguments are $a_1 = 60$, $a_2 = 70$, $a_3 = 40$, $a_4 = 80$. In this case $id(1) = 4$, $id(2) = 2$, $id(3) = 1$, and $id(4) = 3$ hence $W = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.4 \\ 0.1 \end{bmatrix}$ and $B = \begin{bmatrix} 80 \\ 70 \\ 60 \\ 40 \end{bmatrix}$ and we get

$$\begin{aligned} \text{OWA}(a_1, a_2, a_3, a_4) &= [0.30.20.40.1] \begin{bmatrix} 80 \\ 70 \\ 60 \\ 40 \end{bmatrix} \\ &= (0.3)(80) + (0.2)(70) + (0.4)(60) \\ &\quad + (0.1)(40)\text{OWA}(a_1, a_2, a_3, a_4) \\ &= 24 + 14 + 24 + 4 = 66 \end{aligned}$$

We emphasize that there are no restrictions on the numeric arguments in the OWA operator, they can be positive or negative and can be any real number. We note that while in many applications we consider that the argument values are drawn from the unit interval this is not necessary.

At times, when we want to emphasize which OWA weighting vector is being used, we shall use the notation OWA_W to highlight that W is the weighting vector.

If the OWA operator has weighting vector $W = \begin{bmatrix} w_1 \\ w_n \end{bmatrix}$ then the dual OWA operator $\widehat{OWA}$ has weighting vector $\hat{W} = \begin{bmatrix} w_n \\ w_1 \end{bmatrix}$ that is if $\hat{w}_j$ indicates the weights of the dual OWA operator then $\hat{w}_j = w_{n-j+1}$

We note that in the special case when the $a_i \in [0, 1]$ we show below that

$$\text{OWA}_W(a_1, \ldots, a_n) = 1 - \text{OWA}_{\hat{W}}(1 - a_i, \ldots, 1 - a_n)$$

Proof. Let us denote $b_i = 1 - a_i$. Consider $\text{OWA}_{\hat{W}}(b_1, \ldots, b_n)$. We see that $\text{OWA}_{\hat{W}}(b_1, \ldots, b_n) = \sum_{j=1}^{n} w_j \hat{b}_{\,id(j)}$ when $\widehat{id}$ is the ordering function on the b_i. However since the order of the b_i is dual to the order of the a_j we see that $\widehat{id}(j) = n - id(j) + 1$. Here then

$$\begin{aligned} OWA_{\hat{W}}(b_1, \ldots, b_n) &= \sum_{j=1}^{n} \hat{w}_j b_{n-id(j)+1} = \sum_{j=1}^{n} \hat{w}_j (1 - a_{n-id(j)+1}) \\ &= \sum_{j=1}^{n} \hat{w}_j - \sum_{j=1}^{n} \hat{w}_j a_{n-id(j)+1} \\ &= 1 - \sum_{j=1}^{n} w_{j-j+1} a_{n-id(j)+1} \\ &= 1 - \sum_{k=1}^{n} w_k a_{id(k)} = 1 - OWA_W(a_1, \ldots, a_n) \end{aligned}$$

In the case where we restrict ourselves to arguments that are in the unit interval we can introduce various generalization of the OWA operator [10,13].

Generalized OWA Operator. If g: $[0, 1] \to [-\infty, \infty]$ is a continuous strictly monotonic function and W is an OWA weighting vector the function $\text{OWA}(a_1, \ldots, a_n) = g^{-1}(\sum_{j=1}^{n} w_j g(a_{id(j)}))$ is called the generalized OWA operator.

A special case of this is the power-based generalized OWA operator. Here, for any real number $r \in R$ we have

$$\text{Power OWA } (a, \ldots, a) = \left(\sum_{j=1}^{n} w_i (a_{id(j)})^r \right)^{1/r}$$

Another related operator is the ordered weight geometric function

$$\text{OWG}(a_1, \ldots, a_n) = \prod_{j=1}^{n} (a_{id(j)})^{w_j}$$

For the most part in the following we shall focus on the standard OWA operator and also restrict ourselves to the case where the argument values, the $a_i \in [0, 1]$.

The OWA operator provides a parameterized class of aggregation operators that are parameterized by the choice of weighting vector. Different weighting vectors result in different formulation of the OWA operator. Let us look at the formulation of OWA aggregation for some notable examples of weighting vectors

1. The vector W^* has $w_1 = 1$ and $w_j = 0$ for $j \neq 1$. In this case we get $\text{OWA}_{W^*}(a_1, \ldots, a_n) = \text{Max}_i[a_i]$.
2. The vector W_* has $w_n = 1$ and $w_j = 0$ for $j \neq n$. In the case $\text{OWA}_{W_*}(a_1, \ldots, a_n) = \text{Min}_i[a_i]$.

We easily see that these are the bounding weighting vectors and for any weighting vector W

$$\text{OWA}_{W_*}(a_1, \ldots, a) \leq \text{OWA}_W(a_1, \ldots a_n) \leq \text{OWA}_{W^*}(a_1, \ldots, a_n)$$

Closely related to these are the family of vectors $W_{[K]}$ defined such that $w_K = 1$ and $w_j = 0$ for $j \neq K$. In this case we easily see $\text{OWA}_{W_{[K]}}(a_1, \ldots, a_n) = a_{id(K)}$ the kth largest argument. We note that $W^* = W_{[1]}$ and $W^* = W_{[n]}$. Furthermore there is a simple ordering among these $\text{OWA}_{W_{[K]}}$ so if $K_1 \leq K_2$ the $\text{OWA}_{W_{[K_2]}}(a_1, \ldots, a_n) \geq \text{OW A}_{W_{[K_2]}}(a_1, \ldots, a_n)$.

Another special case of this class is the median type operator. Here if n is odd the $\text{Med}(a_1, \ldots, a_n)$ corresponds to the vector $W_{[K]}$ where $K = \frac{n+1}{2}$, If n is even the median uses the vector W_{Med} which has $w_{n/2} = 0.5$ and $w_{n/2} + 1 = 0.5$ and all other $w_j = 0$.

Another important special case is W_A where $w_j = 1/n$ for all j. Here, we get $\text{OWA}_{W_A}[a, \ldots, a) = \frac{1}{n} \sum_{j=1}^{n} a_j$, it is the simple average of the arguments.

Another example of OWA operator is the Arrow–Hurwicz average. Here, W is such that $w_1 = \alpha$ and $w_n = 1 - \alpha$ and all other $w_j = 0$. In this case

$$\text{OWA}_W(a_1, \ldots, a_n) = \alpha \text{Max}_i[a_i] + (1 - \alpha)\text{Min}_i[a_i]$$

In [1] Yager suggested two measures, dependent on the weighted vector, for characterizing the OWA operator. The first of these is called the attitudinal character defined as $\text{AC}(W) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j$. Essentially the attitudinal character provides some information about the preference of the OWA operator for giving more weight to the bigger or smaller argument values. The closer $\text{AC}(W)$ to one the more preference to bigger argument values while values of $\text{AC}(W)$ closer to zero indicate a preference is given the smaller argument values. We note that a value of $\text{AC}(W)$ close to 0.5 indicates the aggregation is neutral with respect to showing