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Noise further expresses exponential decay for globally exponentially stable time-varying delayed neural networks

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ABSTRACT

This paper shows that the globally exponentially stable neural network with time-varying delay and bounded noises may converge faster than those without noise. And the influence of noise on global exponential stability of DNNs was analyzed quantitatively. By comparing the upper bounds of noise intensity with coefficients of global exponential stability, we could deduce that noise is able to further express exponential decay for DNNs. The upper bounds of noise intensity are characterized by solving transcendental equations containing adjustable parameters. In addition, a numerical example is provided to illustrate the theoretical result.

1. Introduction

Neural networks (NNs) are nonlinear dynamic systems with some resemblance to biological neural networks in the brain. The stability of NNs depends mainly on their parametrical configuration. In biological neural systems, signal transmission via synapses is usually a noisy process influenced by random fluctuations from the release of neurotransmitters and other disturbances (Haykin, 1994). Moreover, in the implementation of NNs, external random disturbances and time delays of signal transmission are common and can hardly be avoided. It is known that random disturbances and time delays in the neuron activations may result in oscillation or instability of NNs (Pham, Pakdaman, & Virbert, 1998). The stability properties of delayed NNs (DNNs) and stochastic NNs (SNNs) with external random disturbances have been widely investigated in recent years (see,e.g., Arik, 2002, Cao, Yuan, & Li, 2006, Chen, 2001, Chua & Yang, 1988, Huang, Ho, & Lam, 2005, Huang, Li, Duan, & Starzyk, 2012, Liao & Wang, 2003, Liu & Cao, 2009, 2010, 2011, Shen & Wang, 2007, 2008, 2012, Wang, Liu, Li, & Liu, 2006, Zeng & Wang, 2006, Zeng, Wang, & Liao, 2005, Zhang, Wang, & Liu, 2014, Zhu, Shen, & Chen, 2010 and the references cited therein).

It is well known that noise can be used to stabilize a given unstable system, and it also can make a stable system even more stable. There is an extensive literature concerned with the stabilization by noise, e.g., (Appleby, Mao, & Rodkina, 2008; Luo, Zhong, Zhu, & Shen, 2014; Mao, 2005, 2007b; Zhu & Shen, 2013) and the references therein. The pioneering work in this area is given due to Hasminskii (1981), who stabilized an unstable system by using two white noise sources. Several years ago, Mao, Marion, and Renshaw (2002) showed another important fact that the environmental noise can suppress explosions in a finite time in population dynamics. Recently, Deng, Luo, Mao, and Peng (2008) revealed that the noise can suppress or express exponential growth under the linear growth condition. In Hu, Liu, Mao, and Song (2009), Hu et al. developed the theory in Deng et al. (2008) to cope with the much more general systems. In absence of the linear growth condition or only the one-sided linear growth condition, Wu and Hu (2009) further considered the problem of stochastic suppression and stabilization of nonlinear differential systems. Liu and Shen (2012) revealed that the single noise can also make almost every path of the solution of corresponding stochastically perturbed system grow at most polynomially.

Noises can lead to instability and they can destabilize stable DNNs if it exceeds their limits, what is more, the instability depends on the noise intensity. For a stable DNN, is there a certain noise intensity that can make the DNN even more stable? Therefore, it is interesting to determine the upper bounds of random disturbances which express exponential decay for a stable DNN without losing its global exponential stability. Although the various stability properties of DNNs with noise have been extensively investigated by employing the Lyapunov stability theory (Arik, 2002; Chen, 2001; Chua & Yang, 1988; Liao & Wang, 2003; Shen & Wang,

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2007, 2008), the linear matrix inequality methods (Cao et al., 2006; Huang et al., 2005; Wang et al., 2006; Xu, Lam, & Ho, 2006) and the matrix norm theory (Faydasicok & Arik, 2012), few works investigated the issue: noise makes the DNN even more stable when it is already stable, directly by estimating the upper bounds of noise level from the coefficients of global exponential stability condition.

Motivated by the above discussions, we quantitatively analyze the influence of noise on global exponential stability of DNNs. Different from the traditional Lyapunov stability theory and the matrix norm theory, we investigate the exponential stability of DNNs directly from the coefficients of the DNNs. In this paper, we know that noise is able to further express exponential decay for DNNs without losing global stability, by comparing the upper bounds of noise intensity and coefficients of global exponential stability. The upper bounds of noise intensity are characterized by solving transcendental equations containing adjustable parameters.

2. Problem formulation

Throughout this paper, unless otherwise specified, R^n and $R^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. the filtration contains all P-null sets and is right continuous). $\omega(t)$ be a scalar Brownian motion defined on the probability space. If A is a matrix, its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$, where $|\cdot|$ is the Euclidean norm. Denote $L^2_{\mathscr{F}_0}([-\bar{\tau}, 0]; R^n)$ as the family of all $\mathscr{F}_0-$ measurable $C([-\bar{\tau}, 0]; R^n)$ valued random variables $\psi = \{\psi(\theta) : -\bar{\tau} \leq \theta \leq 0\}$ such that $\sup_{-\bar{\tau}\leq\theta\leq 0} E|\psi(\theta)|^2 < \infty$ where $E\{\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

Consider a DNN model

$$\begin{aligned} dz(t) &= [-Az(t) + Bg(z(t)) + Dg(z(t-\tau(t))) + I]dt, \\ z(t) &= \psi(t-t_0) \in C([t_0-\bar{\tau}, t_0]; R^n), \\ &\quad t_0-\bar{\tau} \leq t \leq t_0, \end{aligned} \tag{1}$$

where $z(t) = (z_1(t), \ldots, z_n(t))^T \in R^n$ is the state vector of the neurons, $t_0 \in R_+$ and $\psi \in R^n$ are the initial values, $A = \mathrm{diag}\{a_1, \ldots, a_n\} \in R^{n\times n}$, $a_i > 0$ is the self-feedback connection weight matrix, $B = (b_{kl})_{n\times n} \in R^{n\times n}$, $D = (d_{kl})_{n\times n} \in R^{n\times n}$ are connection weight matrices, $\tau(t)$ is a delay, which satisfies $\tau(t) : [t_0, +\infty) \to [0, \bar{\tau}]$, $\tau'(t) \leq \mu < 1$, $\psi = \{\psi(s) : -\bar{\tau} \leq s \leq 0\} \in C([-\bar{\tau}, 0], R^n)$, $\bar{\tau}$ is the maximum of delay, I is neuron external input (bias), and $g(\cdot) \in R^n$ is a continuous bounded vector-valued activation function which satisfying the following Lipschitz condition; i.e.,

$$|g(u) - g(v)| \leq k|u - v|, \quad \forall u, v \in R^n,\ g(0) = 0,$$

where k is a known constant.

As usual, a vector $z^* = [z_1^*, z_2^*, \ldots, z_n^*]^T$ is said to be an equilibrium point of system (1) if it satisfies

$$Az^* = (B + D)g(z^*) + I$$

For notational convenience, we will always shift an intended equilibrium point z^* of system (1) to the origin by letting $x = z - z^*$, $f(x) = g(x + z^*) - g(z^*)$. It is easy to transform system (1) into the following form:

$$\begin{aligned} dx(t) &= [-Ax(t) + Bf(x(t)) + Df(x(t-\tau(t)))]dt, \\ x(t) &= \psi(t-t_0) \in C([t_0-\bar{\tau}, t_0]; R^n), \\ &\quad t_0-\bar{\tau} \leq t \leq t_0. \end{aligned} \tag{2}$$

In addition, the function f in (2) satisfies the following Lipschitz condition and $f(0) = 0$:

Assumption 1. The activation function $f(\cdot)$ satisfies the following Lipschitz condition; i.e.,

$$|f(u) - f(v)| \leq k|u - v|, \quad \forall u, v \in R^n,\ f(0) = 0, \tag{3}$$

where k is a known constant.

DNN (2) has a unique state $x(t; t_0, \psi)$ on $t \geq t_0$ for any initial value t_0, ψ. Now we define the globally exponential stability of the state of DNN (2).

Definition 1. The state of DNN (2) is globally exponentially stable, if for any t_0, ψ, there exist $\alpha > 0$ and $\beta > 0$ such that

$$|x(t; t_0, \psi)| \leq \alpha\|\psi\| \exp(-\beta(t - t_0)), \quad \forall t \geq t_0, \tag{4}$$

where $x(t; t_0, \psi)$ is the state of the model in (2).

3. Main results

Now, the question is, for a given globally exponentially stable DNN, how much noise intensity can the DNN endure without impacting its stability? We consider the noise-induced DNNs described by the Itô stochastic differential equation (SDNNs).

$$\begin{aligned} dy(t) &= [-Ay(t) + Bf(y(t)) + Df(y(t-\tau(t)))]dt \\ &\quad + \sigma y(t)d\omega(t), \quad t > t_0, \\ y(t) &= \psi(t-t_0) \in L^2_{\mathscr{F}_0}([t_0-\bar{\tau}, t_0]; R^n), \\ &\quad t_0-\bar{\tau} \leq t \leq t_0, \end{aligned} \tag{5}$$

where A, B, D, f are the same as in Section 2, f satisfies Assumption 1, σ is the intensity of noise, $\omega(t)$ is a scalar Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$.

Under Assumption 1, SDNN (5) has a unique state for any initial value t_0, ψ and the origin point is the equilibrium point. We calculated the largest noise intensity that SDNN (5) can bear without losing global exponential stability. Moreover, we deduce the intensity of noise that makes the DNN (2) even more stable when it is already stable. For SDNN (5), we give the following definition of global exponential stability.

Definition 2 (*Mao, 2007a*)**.** SDNN (5) is said to be almost surely globally exponentially stable if for any $t_0 \in R_+, \psi \in L^2_{\mathscr{F}_0}([-\bar{\tau}, 0]; R^n)$, there exist $\alpha > 0$ and $\beta > 0$ such that $\forall t \geq t_0$, $|y(t; t_0, \psi)| \leq \alpha\|\psi\| \exp(-\beta(t - t_0))$ hold almost surely; i.e., the Lyapunov exponent $\limsup_{t\to\infty} (\ln|y(t; t_0, \psi)|/t) < 0$ almost surely, where $y(t; t_0, \psi)$ is the state of SDNN (5). SDNN (5) is said to be mean square globally exponentially stable if for any $t_0 \in R_+$, $\psi \in L^2_{\mathscr{F}_0}([-\bar{\tau}, 0]; R^n)$, there exist $\alpha > 0$ and $\beta > 0$ such that $\forall t \geq t_0$, $E|y(t; t_0, \psi)|^2 \leq \alpha\|\psi\| \exp(-\beta(t - t_0))$ hold; i.e., $\limsup_{t\to\infty}(\ln(E|y(t; t_0, \psi)|^2)/t) < 0$, where $y(t; t_0, \psi)$ is the state of SDNN (5).

From the above definitions, the almost surely global exponential stability and the mean square global exponential stability of SDNN (5) are formally corresponding to each other. In fact, they do not imply each other and additional conditions are required in order to deduce one from the other. Therefore, if Assumption 1 holds, we have the following lemma (Mao, 2007a).

Lemma 1. *Let Assumption 1 hold. Then the global exponential stability in sense of mean square of SDNN (5) implies the almost surely exponential stability of SDNN (5).*

Theorem 1. *Let Assumption 1 hold and DNN (2) be globally exponentially stable when the coefficient of global exponential stability $\alpha < \sqrt{2}/2$. SDNN (5) is mean square globally exponentially*