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Applications of Laplacian spectra for n-prism networks

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ABSTRACT

In this paper, the properties of the Laplacian matrices for the n-prism networks are investigated. We calculate the Laplacian spectra of n-prism graphs which are both planar and polyhedral. In particular, we derive the analytical expressions for the product and the sum of the reciprocals of all nonzero Laplacian eigenvalues. Moreover, these results are used to handle various problems that often arise in the study of networks including Kirchhoff index, global mean-first passage time, average path length and the number of spanning trees. These consequences improve and extend the earlier results.

1. Introduction

In recent decades, complex networks have been widely investigated since they have been proven as a powerful tool to describe and characterize lots of complex systems in nature and society [1,2]. The study on complex networks is comprehensive and challenging, which involves the subjects of nonlinear dynamics, control theory and graph theory [2,3].

Many networks are usually modeled as undirected graphs, in which vertices and edges correspond to processors and communication links, respectively. Let G be a graph with vertices labelled as $1, 2, \dots, n$. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix with (i,j)-entry equal to 1 if vertices i and j are adjacent and 0 otherwise. The eigenvalues of a graph G are the eigenvalues of $A(G)$. Let $D(G) = diag[d_1, d_2, \dots, d_n]$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$, denoted by $\mu_1 \le \mu_2 \le \cdots \le \mu_n$, are called the Laplacian spectra of G. It is well known that $\mu_1 = 0$. Particularly, $\mu_2 > 0$ if and only if G is a connected graph [4].

Laplacian spectra and their applications are involved in diverse theoretical problems on complex networks [3,5,6]. Many results have been devoted to studying Laplacian spectra for complex networks [7,8,10]. Calculating the Laplacian spectra of networks has many applications in lots of aspects, such as the topological structures and dynamical processes [11,12]. For instance, the diameter of a network can be defined by the second smallest eigenvalue of Laplacian matrix. The number of spanning trees of networks can be determined by the product of all nonzero Laplacian eigenvalues, and the Kirchhoff index of networks can be expressed by the sum of reciprocals of nonzero eigenvalues [2,3]. Additionally, the synchronizability of a network is determined by the ratio of the maximum eigenvalue to the smallest nonzero one of its Laplacian matrix [13,14]. Consequently, it is of interest to formulate the Laplacian spectra of networks, though determining this analytically is a theoretical challenge.

The applications and Laplacian spectra of extended Koch networks have been studied in [3]. The related problems of the Laplacian spectra of a 3-prism network were also obtained in [2]. Motivated by the above-mentioned results, we consider the work of computation of Laplacian spectra for n-prism networks. In this paper, we calculate the Laplacian spectra of an n-prism network which is widely studied in the subject of graph theory. The exact analytical expressions for the product and the sum of the reciprocals of all nonzero Laplacian eigenvalues are derived. By applying these expressions to calculate various problems, we improve and extend the earlier results.

The rest of this paper is organized as follows. Section 2 introduces the construction of the n-prism networks. Calculations of Laplacian spectra and its applications are proposed in Sections 3 and 4, respectively. Finally conclusions are included in Section 5.

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2. Structure constructions of the *n*-prism networks

An n-prism network is built in an iterative way. Let $P(g)$ $(g \geq 1)$ be the family of this graph after $g-1$ iterations. Initially at $g=1$, $P(1)$ is an n-polygon. For $g \geq 2, P(g)$ is built from $P(g-1)$, where every existing node in $P(g-1)$ gives birth to a new node and the n new nodes form a new n-polygon, then each new node is also connected to its corresponding "mother" node, which is illustrated by Fig. 1. Fig. 2 shows the structure characteristic of n-prism network $P(g)$. It is easy to see that the number of nodes and edges in $P(g)$ is $N_g = ng$ and $E_g = (2g-1)n$, respectively.

3. Calculations of the Laplacian spectra

In this section, we will calculate the Laplacian eigenvalues for n-prism networks. We first introduce a definition and some lemmas which will be used later on.

Definition 1. The Kronecker product $A \otimes B$ of two matrices A and B is the matrix obtained by replacing the (i,j)-entry a_{ij} of A by $a_{ij}B$, for all i and j [15].

The Kronecker product has the following properties [16].

Lemma 3.1. *Let* $A \in M_{m,n}(\mathbb{F}), B \in M_{p,q}(\mathbb{F}), C \in M_{n,k}(\mathbb{F}), D \in M_{q,r}(\mathbb{F})$ *and* $\alpha \in \mathbb{F}$.

Then 1. $(A \otimes B)^T = A^T \otimes B^T$.

2. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
3. $\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B)$.

Let P_n and C_n denote the path and cycle with n vertices, respectively. Then their spectra can be stated (see [4, p. 18]) as:

Lemma 3.2. *The Laplacian eigenvalues of a path* P_n *are* $2-2\cos\frac{i\pi}{n}$ $(i=0,1,\ldots,n-1)$, *and the Laplacian eigenvalues of a cycle* C_n *are* $2-2\cos\frac{2\pi j}{n}$ $(j=0,1,\ldots,n-1)$.

Let A_g and B_g be the product of all nonzero eigenvalues of $L(\mathbf{P}_g)$ and the sum of the reciprocals of these eigenvalues, respectively. That is

$$A_g = \prod_{k=2}^{N_g} \mu_k$$

and

$$B_g = \sum_{k=2}^{N_g} \frac{1}{\mu_k},$$

where μ_k $(k=2,3,\ldots,N_g)$ denote the N_g-1 nonzero eigenvalues of $L(\mathbf{P}_g)$ and $\mu_1 = 0$.

Theorem 3.3. *The product and sum of reciprocal nonzero eigenvalues of* $L(\mathbf{P}_g)$ *are*

1. $A_g = \prod_{i=0}^{g-1}\prod_{j=0}^{n-1}\left(4-2\cos\frac{i\pi}{g}-2\cos\frac{2j\pi}{n}\right)$,
   $i=0,1,\ldots,g-1; j=0,1,\ldots,n-1; (i,j) \neq (0,0)$.
2. $B_g = \sum_{i=0}^{g-1}\sum_{j=0}^{n-1}\frac{1}{\left(4-2\cos\frac{i\pi}{g}-2\cos\frac{2j\pi}{n}\right)}$,
   $i=0,1,\ldots,g-1; j=0,1,\ldots,n-1; (i,j) \neq (0,0)$.

Proof. With a suitable labeling for nodes of n-prism network and by Lemma 3.1, the Laplacian matrix can be written as

$$L(\mathbf{P}_g) = L(C_n) \otimes \mathbf{I}_g + \mathbf{I}_n \otimes L(P_g),$$

where $\mathbf{I}_n$ denotes the identity matrix of dimension $n \times n$.

Actually, there exists invertible matrices P, Q such that the matrices

$$(L(C_n))' := P^{-1}L(C_n)P, \quad (L(P_g))' := Q^{-1}L(P_g)Q$$

are both upper triangular with diagonal elements

$2-2\cos\frac{2\pi j}{n}$ $(j=0,1,\ldots,n-1)$ and $2-2\cos\frac{i\pi}{n}$ $(i=0,1,\ldots,g-1)$, respectively.

And it is easily seen that

$$(P \otimes Q)^{-1} \cdot \left(L(C_n) \otimes \mathbf{I}_g + \mathbf{I}_n \otimes L(P_g)\right) \cdot (P \otimes Q) = L(C_n)' \otimes \mathbf{I}_g + \mathbf{I}_n \otimes L(P_g)'$$

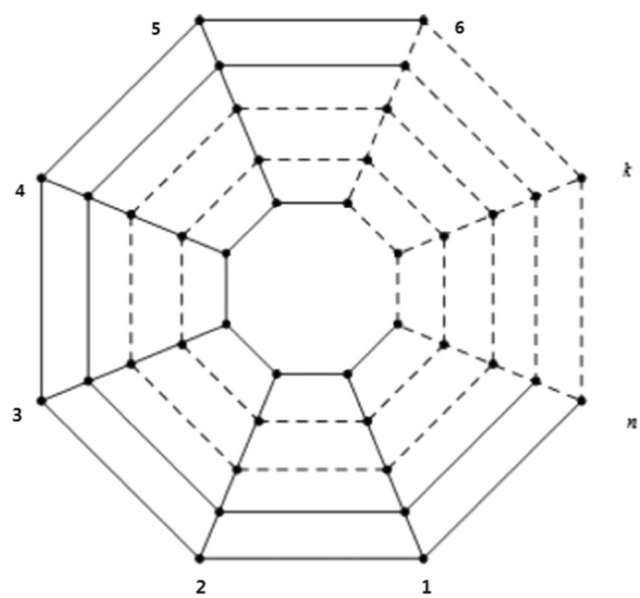

Fig. 2. On the basis of Fig. 1, g-generation n-prism network mainly consists of $(g-1)$-generation prism network, which extends the situation of $g=1$.

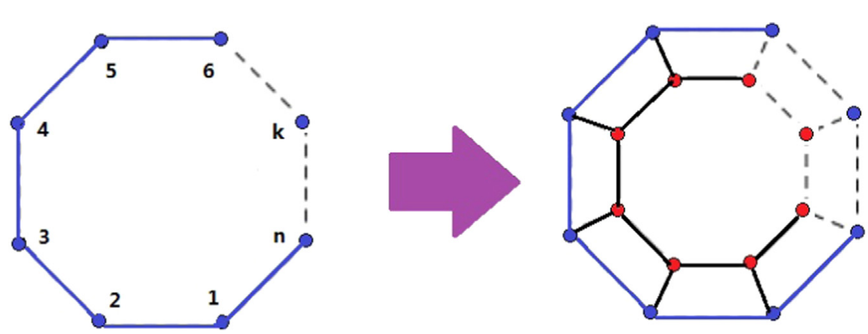

Fig. 1. A way of construction for the n-prism networks.