Computational Geometry 53 (2016) 36–50

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Geometric permutations of non-overlapping unit balls revisited ☆

Jae-Soon Ha [a], Otfried Cheong [a,*], Xavier Goaoc [b], Jungwoo Yang [c]

[a] KAIST, Republic of Korea
[b] Université Paris-Est Marne la Vallée, France
[c] Aarhus University, Denmark

ARTICLE INFO

Article history:
Received 17 February 2015
Accepted 20 December 2015
Available online 29 December 2015

Keywords:
Transversal theory
Line transversal
Unit ball
Congruent balls
Geometric permutation

ABSTRACT

Given four congruent balls A, B, C, D in $\mathbb{R}^\delta$ that have disjoint interior and admit a line that intersects them in the order $ABCD$, we show that the distance between the centers of consecutive balls is smaller than the distance between the centers of A and D. This allows us to give a new short proof that n interior-disjoint congruent balls admit at most three geometric permutations, two if $n \geqslant 7$. We also make a conjecture that would imply that $n \geqslant 4$ such balls admit at most two geometric permutations, and show that if the conjecture is false, then there is a counter-example that is algebraically highly degenerate.

1. Introduction

A *line transversal* to a family $\mathfrak{F}$ of pairwise disjoint convex sets in $\mathbb{R}^\delta$ is a line that intersects every element of that family. The study of line transversals, their properties, and conditions for their existence started in the 1950s with the classic work of Grünbaum, Hadwiger, and Danzer; background about the sizable literature on *geometric transversal theory* can be found in the classic survey of Danzer et al. [1], or the more recent ones by Goodman et al. [2], Eckhoff [3], Wenger [4], or Holmsen [5].

An oriented line transversal ℓ to a family $\mathfrak{F}$ induces a linear order on $\mathfrak{F}$: Fig. 1(a) shows three oriented transversals to a family of three congruent disks inducing the orders $A \prec C \prec B$, $A \prec B \prec C$, and $B \prec A \prec C$. For conciseness, we usually represent the order by the string listing the elements, the three lines in Fig. 1(a) induce the orders *ACB*, *ABC*, and *BAC*. Natural questions in geometric transversal theory are: Given a family of disjoint convex objects, how many different orders can be realized by line transversals? How much can these orders differ? What becomes of these questions if the objects have a more restricted shape, for instance if they are balls or axis-aligned boxes?

If an order can be realized by an oriented line, so can its reverse and the two orders are therefore equivalent as far as line transversals are concerned. The equivalence classes, that is, pairs of an order and its reverse, are called *geometric permutations*. Fig. 1(b) shows a set of five congruent disks with the two geometric permutations *ABCDE* and *ACBDE*, which could equally well be written as *EDCBA* and *EDBCA*. In Fig. 1(b) the disks B and C touch each other. We allow this, *but a line transversal is not allowed to be tangent to these disks in this common point*. Put differently, we can remove the common points of

☆ This research was supported in part by NRF grant 2011-0030044 (SRC-GAIA) from the government of Korea, and in part by ICT R&D program of MSIP/IITP [R0126-15-1108].

* Corresponding author.
E-mail addresses: jaesoonha@kaist.ac.kr (J.-S. Ha), otfried@kaist.edu (O. Cheong), goaoc@u-pem.fr (X. Goaoc), jungwoo@madalgo.au.dk (J. Yang).

http://dx.doi.org/10.1016/j.comgeo.2015.12.003
0925-7721/

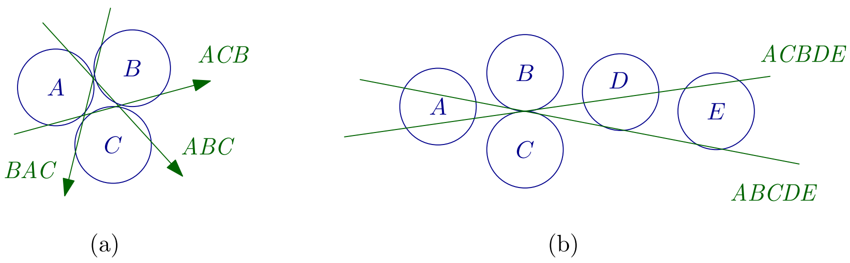

Fig. 1. Orders and geometric permutations.

contact from the objects to obtain a family of disjoint convex objects with the same set of line transversals. It is convenient to allow such families, as configurations are often easier to describe when objects touch. We will call a family of compact convex objects in $\mathbb{R}^\delta$ that may touch, but whose interior is disjoint, a *non-overlapping* family.

The study of geometric permutations started in the 1980s with the work by Katchalski et al. [6,7]. In the plane, n convex objects admit at most $2n-2$ geometric permutations and this bound is tight [8]. One of the intriguing open questions is the corresponding bound for three and higher dimensions: n convex objects in $\mathbb{R}^\delta$ can have $\Omega(n^{\delta-1})$ geometric permutations [9], but the best known upper bound is only $O(n^{2\delta-3}\log n)$ [10]. For balls or similar *fat* objects, the lower bound of $\Omega(n^{\delta-1})$ is known to be tight [9,11]. Disjoint *congruent* balls, however, have only a constant number of geometric permutations: In two dimensions, $n \geqslant 4$ congruent disks have at most two geometric permutations [9,12]. In dimension $\delta \geqslant 3$, Cheong et al. [13] proved that n non-overlapping congruent balls have at most three geometric permutations, and at most two geometric permutations when $n \geqslant 9$.

In this paper we revisit the problem of bounding the number of geometric permutations of n non-overlapping congruent balls in $\mathbb{R}^\delta$. Since we can arbitrarily choose the radius of the balls, we will refer to them as *unit balls*. The earlier work of Cheong et al. [13] does not entirely settle the question, as no construction of $n > 3$ non-overlapping unit balls is known that admits more than two geometric permutations. Furthermore, the proof by Cheong et al. is quite technical and relies on delicate geometric lemmas and tedious case analysis.

In the first part of this paper, we give a shorter and greatly simplified proof that $n \geqslant 3$ non-overlapping unit balls have at most three geometric permutations. Unlike the previous proof [13], it could be presented in its entirety in an undergraduate course on transversal theory. Our main theorem is the following:

Theorem 1. *Let $\mathfrak{F}$ be a family of n non-overlapping unit balls in $\mathbb{R}^\delta$. The number of geometric permutations of $\mathfrak{F}$ is at most three if $n \leqslant 6$, and at most two if $n \geqslant 7$.*

Theorem 1 slightly improves the previous bound of Cheong et al. [13] by giving a tight bound for $n = 7$ and 8 (so that only for $4 \leqslant n \leqslant 6$ it remains open whether the number of geometric permutations is two or three). Our proof relies on the following lemma:

Distance Lemma. *If four non-overlapping unit balls A, B, C and D in $\mathbb{R}^\delta$ have a line transversal with the order $ABCD$ then $|ad| > \max\{|ab|, |bc|, |cd|\}$.*

(Here and throughout the paper we will use lower-case letters to denote the centers of balls written with upper-case letters, so a, b, c, and d are the centers of A, B, C, and D.) The lemma is not as obvious as it might appear and is false for three balls: Fig. 1(a) shows that $|ac| < |ab|$ is possible for three unit balls with a transversal with the order ABC.

We prove the Distance Lemma, in Section 3, by first modifying the given configuration into a canonical situation: We shrink the balls, keeping them congruent, until we reach the smallest radius for which they still have a transversal with the given order. This idea has probably been used first by Klee [14] and then by Hadwiger [15]. The resulting canonical configuration $\mathfrak{F}$ has the property that the line transversal ℓ is *pinned* (Lemma 7): This means that any arbitrarily small perturbation of ℓ is no longer a transversal of $\mathfrak{F}$. In other words, ℓ is an isolated point in the space of transversals of $\mathfrak{F}$. The same method for deforming a family of unit balls such that the line transversal becomes pinned has been used by Cheong et al. [16]. The correctness of the method is there deduced from algebraic results by Megyesi and Sottile [17] and by Borcea et al. [18]. This argument requires strict disjointness of the balls, and doesn't meet our goal of a proof presentable to undergraduates. We instead observe that the fact we need is already implicit in a proof by Holmsen et al. [19]. In Appendix A we examine their proof to prove the correctness of the pinning method for non-overlapping unit balls.

Before proving the Distance Lemma, we show, in Section 2, that it readily simplifies various steps of the proof of Cheong et al. [13], resulting in an elementary proof that the number of geometric permutations of n non-overlapping unit balls is at most three. On the one hand, the Distance Lemma simplifies technical derivations. For example, the fact that the geometric permutations $ABCD$ and $BADC$ are *incompatible* for non-overlapping unit balls, that is, they cannot be realized at the same time by a family of four balls, was given a delicate, five pages long, proof [13, Section 4]; it follows immediately from