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The non-pure version of the simplex and the boundary of the simplex

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ABSTRACT

We introduce the non-pure versions of simplicial balls and spheres with minimum number of vertices. These are a special type of non-homogeneous balls and spheres (NH-balls and NH-spheres) satisfying a minimality condition on the number of facets. The main result is that *minimal* NH-balls and NH-spheres are precisely the simplicial complexes whose iterated Alexander duals converge respectively to a simplex or the boundary of a simplex.

1. Introduction

A simplicial complex K of dimension d is *vertex-minimal* if it is a d-simplex or it has $d+2$ vertices. It is not hard to see that a vertex-minimal homogeneous (or pure) complex of dimension d is either an elementary starring $(\tau, a)\Delta^d$ of a d-simplex or the boundary $\partial\Delta^{d+1}$ of a $(d+1)$-simplex. On the other hand, a general non-pure complex with minimum number of vertices has no precise characterization. However, since vertex-minimal pure complexes are either balls or spheres, it is natural to ask whether there is a non-pure analogue to these polyhedra within the theory of non-homogeneous balls and spheres. In [5] G. Minian and the author introduced NH-manifolds, a generalization of the concept of manifold to the non-pure setting (somewhat similar to Björner and Wachs's extension of the shellability definition to non-pure complexes [3]). In this theory, NH-balls and NH-spheres are the non-pure versions of combinatorial balls and spheres.

The purpose of this article is to study *minimal* NH-balls and NH-spheres, which are respectively the non-pure counterpart of vertex-minimal balls and spheres. Note that $\partial\Delta^{d+1}$ is not only the d-sphere with minimum number of vertices but also the one with minimum number of facets. For non-pure spheres, this last property is strictly stronger than vertex-minimality and it is convenient to define minimal NH-spheres as the ones with minimum number of facets. With this definition, minimal NH-spheres with the homotopy type of a k-sphere are precisely the non-pure spheres whose nerve is $\partial\Delta^{k+1}$, a property that also characterizes the boundary of simplices. On the other hand, an NH-ball B is minimal if it is part of a decomposition of a minimal NH-sphere, i.e. if there exists a combinatorial ball L with $B \cap L = \partial L$ such that $B + L$ is a minimal NH-sphere. This definition is consistent with the notion of vertex-minimal simplicial ball (see Lemma 4.1 below).

Surprisingly, minimal NH-balls and NH-spheres can be characterized by a property involving Alexander duals. Denote by K^* the Alexander dual of a complex K relative to the vertices of K. Set inductively $K^{*(0)} = K$ and $K^{*(m)} = (K^{*(m-1)})^*$. Thus, in each step $K^{*(i)}$ is computed relatively to its own vertices, i.e. as a subcomplex of the boundary of the simplex of minimum dimension containing it. We call $(K^{*(m)})_{m\in\mathbb{N}_0}$ the *sequence of iterated Alexander duals* of K. The main result of the article is the following.

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Theorem 1.1. *Let K be a finite simplicial complex.*

(*i*) *There is an $m \in \mathbb{N}_0$ such that $K^{*(m)}$ is the boundary of a simplex if and only if K is a minimal NH-sphere.*
(*ii*) *There is an $m \in \mathbb{N}_0$ such that $K^{*(m)}$ is a simplex if and only if K is a minimal NH-ball.*

In any case, the number of iterations needed to reach the simplex or the boundary of the simplex is bounded above by the number of vertices of K.

Note that $K^* = \Delta^d$ if and only if K is a vertex-minimal d-ball which is not a simplex, so (*ii*) describes precisely all complexes *converging* to vertex-minimal balls. Theorem 1.1 characterizes the classes of Δ^d and $\partial\Delta^d$ in the equivalence relation generated by $K \sim K^*$.

2. Preliminaries

2.1. Notation and definitions

All simplicial complexes that we deal with are assumed to be finite. Given a set of vertices V, $|V|$ will denote its cardinality and $\Delta(V)$ the simplex spanned by its vertices. $\Delta^d := \Delta(\{0, \dots, d\})$ will denote a generic d-simplex and $\partial\Delta^d$ its boundary. The set of vertices of a complex K will be denoted V_K and we set $\Delta_K := \Delta(V_K)$. A *facet* of a complex K is a simplex which is not a proper face of any other simplex of K. We denote by $\mathsf{f}(K)$ the number of facets in K. A *ridge* is a maximal proper face of a facet. A complex is *pure* or *homogeneous* if all its facets have the same dimension.

We denote by $\sigma * \tau$ the join of the faces $\sigma, \tau \in K$ (if $\sigma \cap \tau = \emptyset$) and by $K * L$ the join of the complexes K and L (if $V_K \cap V_L = \emptyset$). By convention, if $\emptyset$ is the empty simplex and $\{\emptyset\}$ the complex containing only the empty simplex then $K * \{\emptyset\} = K$ and $K * \emptyset = \emptyset$. Note that $\partial\Delta^0 = \{\emptyset\}$. For $\sigma \in K$, $lk(\sigma, K) = \{\tau \in K : \ \tau \cap \sigma = \emptyset, \ \tau * \sigma \in K\}$ denotes its *link* and $st(\sigma, K) = \sigma * lk(\sigma, K)$ its *star*. The union of two complexes K, L will be denoted by $K + L$. A subcomplex $L \subseteq K$ is said to be *top generated* if every facet of L is also a facet of K.

The notation $K \searrow L$ will mean that K (simplicially) collapses to L. A complex is *collapsible* if it collapses to a single vertex and *PL-collapsible* if it has a subdivision which is collapsible. The *simplicial nerve* $\mathcal{N}(K)$ of K is the complex whose vertices are the facets of K and whose simplices are the finite subsets of facets of K with non-empty intersection.

Two complexes are *PL-isomorphic* if they have a common subdivision. A *combinatorial d-ball* is a complex *PL*-isomorphic to Δ^d. A *combinatorial d-sphere* is a complex *PL*-isomorphic to $\partial\Delta^{d+1}$. By convention, $\partial\Delta^0 = \{\emptyset\}$ is a sphere of dimension -1. A *combinatorial d-manifold* is a complex M such that $lk(v, M)$ is a combinatorial $(d-1)$-ball or $(d-1)$-sphere for every $v \in V_M$. A $(d-1)$-simplex in a combinatorial d-manifold M is a face of at most two d-simplices of M and the boundary ∂M is the complex generated by the $(d-1)$-simplices which are faces of exactly one d-simplex. Combinatorial d-balls and d-spheres are combinatorial d-manifolds. The boundary of a combinatorial d-ball is a combinatorial $(d-1)$-sphere.

2.2. Non-homogeneous balls and spheres

In order to make the presentation self-contained, we recall first the definition and some basic properties of non-homogeneous balls and spheres. For a comprehensive exposition of the subject, the reader is referred to [5] (see also [6, §2.2] for a brief summary).

NH-balls and NH-spheres are special types of NH-manifolds, which are the non-pure versions of combinatorial manifolds. NH-manifolds have a local structure consisting of regularly-assembled pieces of Euclidean spaces of different dimensions. In Fig. 1 we show some examples of NH-manifolds and their underlying spaces. NH-manifolds, NH-balls and NH-spheres are defined as follows.

Definition. An *NH-manifold* (resp. *NH-ball*, *NH-sphere*) of dimension 0 is a combinatorial manifold (resp. ball, sphere) of dimension 0. An NH-sphere of dimension -1 is, by convention, the complex $\{\emptyset\}$. For $d \geq 1$, we define by induction:

- An *NH-manifold* of dimension d is a complex M of dimension d such that $lk(v, M)$ is an NH-ball or an NH-sphere (possibly of dimension -1) for all $v \in V_M$.
- An *NH-ball* of dimension d is a PL-collapsible NH-manifold of dimension d.
- An *NH-sphere* of dimension d and *homotopy dimension* k is an NH-manifold S of dimension d such that there exist a top generated NH-ball B of dimension d and a top generated combinatorial k-ball L such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B + L$ is a *decomposition* of S and write $\dim_h(S)$ for the homotopy dimension of S.

The definitions of NH-ball and NH-sphere are motivated by the classical theorems of Whitehead [9] and Newman [7] (see e.g. [8, Corollaries 3.28 and 3.13]). Just like for classical combinatorial manifolds, it can be seen that the class of NH-manifolds (resp. NH-balls, NH-spheres) is closed under subdivision and that the link of *every* simplex in an NH-manifold is an NH-ball or an NH-sphere. Also, the homogeneous NH-manifolds (resp. NH-balls, NH-spheres) are