# An ABC-Problem for location and consensus functions on graphs 

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#### Abstract

A location problem can often be phrased as a consensus problem. The median function Med is a location/consensus function on a connected graph $G$ that has the finite sequences of vertices of $G$ as input. For each such sequence $\pi$, Med returns the set of vertices that minimize the distance sum to the elements of $\pi$. The median function satisfies three intuitively clear axioms: (A) Anonymity, (B) Betweenness and (C) Consistency. Mulder and Novick showed in 2013 that on median graphs these three axioms actually characterize Med. This result raises a number of questions: (i) On what other classes of graphs is Med characterized by (A), (B) and (C)? (ii) If some class of graphs has other $A B C$-functions besides Med, can we determine additional axioms that are needed to characterize Med? (iii) In the latter case, can we find characterizations of other functions that satisfy (A), (B) and (C)?

We call these questions, and related ones, the ABC-Problem for consensus functions on graphs. In this paper we present first results. We construct a non-trivial class different from the median graphs, on which the median function is the unique "ABC-function". For the second and third question we focus on $K_{n}$ with $n \geq 3$. We construct various non-trivial $A B C$-functions amongst which is an infinite family on $K_{3}$. For some nice families we present a full axiomatic characterization.


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## 1. Introducing the ABC-Problem

The notions of location and consensus can often be considered the same formally. To illustrate this, suppose $G=(V, E)$ is a finite connected graph and $\{1, \ldots, k\}$ is a set of clients (voters). Each client $i$ selects a most suitable, or preferred, location $x_{i}$ in $V$, and it is the task of a location (consensus) function to return those vertices that best satisfy various constraints and properties deemed appropriate for the particular problem at hand. In location problems the constraints are usually in the form of optimizing certain distance criteria. In consensus problems one usually requires certain simple and acceptable rules or axioms that make the voting a reasonable and rational procedure.

[^0]One of the early papers on location problems is the classical paper of Witzgall in 1965 [28]. Since then hundreds of papers have been written about location problems on graphs using the geodesic metric: for example see the reference lists in [22-26]. The earliest paper on the axiomatic study of consensus is the classical paper of Arrow in 1951 [1]. This was the beginning of a fruitful and rich area of research, see for example [2,3,5-7,24]. Holzman [9] was the first to study a location function axiomatically as a consensus problem, thus combining the areas of location and consensus. For some recent work in this area, see [12-14,16,17,27].

A typical location problem consists of finding a median of a set of clients on a connected graph $G$, where a median is a vertex that minimizes the distance sum to the clients. It is usually modeled as a consensus function Med that returns the set of all medians. In 1996 Vohra [27] characterized the median function axiomatically on tree networks (the continuous version of a tree, where internal vertices of edges are also possible locations). In 1998 McMorris, Mulder and Roberts [15] handled the discrete case. They were able to characterize Med on cube-free median graphs using the three simple axioms Anonymity $(A)$, Betweenness ( $B$ ) and Consistency ( $C$ ). For the general case of arbitrary median graphs they needed an extra axiom, but it was not clear then whether this extra axiom was necessary. Median graphs are a natural common generalization of trees and hypercubes [20]. They are defined by the property that, for any three vertices $u, v, w$, there is a unique vertex that lies simultaneously on a shortest path between each pair of $u, v, w$. At first sight these graphs seem to be quite esoteric, but in [10] a one-to-one correspondence was established between the connected triangle-free graphs and a special subclass of the median graphs. Since median graphs are triangle-free and connected, this implies that, in the universe of all graphs, there are as many median graphs as there are connected triangle-free graphs.

Recently Mulder and Novick [21] settled the unclarity in [15]: the three basic axioms ( $A$ ), (B) and ( $C$ ) actually characterize the median function on any median graph. Calling a consensus function on a connected graph satisfying (A), (B), and (C) an $A B C$-function, the Mulder-Novick result motivates the following problems.
(i) Determine the classes of graphs, on which Med is the unique $A B C$-function.
(ii) If $g$ is a class of graphs admitting other $A B C$-functions besides Med, then determine additional axioms that are needed to characterize Med on $g$.
(iii) On such a class of graphs $q$, study the other $A B C$-functions, and, if possible, characterize these axiomatically by additional axioms.
This set of problems, and related questions on $A B C$-functions, is what we call the $A B C$-Problem for location and consensus functions on graphs.

The aim of this paper is to provide first answers to these questions. In Section 2 we set the stage. In Section 3 we formally define our three basic axioms ( $A$ ), (B) and ( $C$ ), and establish for what profiles the output is already determined by these axioms. In Section 4 we present a non-trivial example of a class of graphs, on which the median function is the unique $A B C$-function. For the second and third question we focus on the complete graphs with $n \geq 3$ vertices. On $K_{n}$ any consensus function is basically a voting procedure on $n$ alternatives, on which there exists a vast literature. Our perspective here is slightly different, because we consider only voting procedures that satisfy $(A),(B)$ and (C). But these still make eminent sense from the point of view of voting theory. For some we have nice characterizations, for others we have first results. What all this shows is that, on $K_{n}$ with $n \geq 3, A B C$-functions are abundant. Moreover, we even have an infinite family on $K_{3}$. Of course all this amounts to only first steps in the study of the $A B C$-Problem for location and consensus functions on graphs.

## 2. Setting the stage

The process of finding an optimal location or reaching consensus about a certain issue can often be phrased as a voting procedure. For many types of voting procedures it is advantageous to consider the population of voters to be ordered. Hence we number the voters. We represent the alternatives on which the voters may cast their votes as vertices in a graph. Usually we equate the voter with the candidate on which the voter casts her/his vote, that is, we list the voters/votes as a sequence of vertices in the graph, and we call such a sequence a profile.

The voting process is represented by a consensus function that assigns to each profile a nonempty set of vertices of the graph. A reasonable voting procedure should follow some rules, and these rules can be phrased as axioms for the associated consensus function. We want these axioms to be appealing and as simple as possible. We now formalize these ideas.

Let $G=(V, E)$ be a finite connected graph with vertex set $V$. A profile of length $k$ on $V$ is a nonempty sequence $\pi=\left(x_{1}, x_{2}, \ldots, x_{k}\right)$. Note that vertices may occur more than once in a profile. We denote the length of $\pi$ by $k=|\pi|$. We call $x_{1}, \ldots, x_{k}$ the elements of $\pi$. A vertex of $\pi$ is a vertex that occurs as an element in $\pi$. The carrier set $\{\pi\}$ of $\pi$ is the set of vertices that occur in $\pi$. So, if a vertex occurs more than once as an element in $\pi$, then we have $|\{\pi\}|<|\pi|$. A subprofile of $\pi$ is just a subsequence of $\pi$. For convenience we allow a subprofile to be empty. We denote by $V^{*}$ the set of all profiles of finite length on $V$, and by $2^{V}-\emptyset$ the family of all nonempty subsets of $V$. A consensus function on a graph $G=(V, E)$ is a function $L: V^{*} \rightarrow 2^{V}-\emptyset$. For convenience, we will write $L\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ instead of $L\left(\left(x_{1}, x_{2}, \ldots, x_{k}\right)\right)$, for any profile $\pi=\left(x_{1}, x_{2}, \ldots, x_{k}\right)$. One of the objectives in the theory of consensus is the axiomatic characterization of consensus functions.

The distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest $u, v$-path or $u, v$-geodesic. Let $\pi=\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ be a profile on $V$. For a vertex $v$ of $G$, we write $D(v, \pi)=\sum_{i=1}^{k} d\left(v, x_{i}\right)$. A vertex $x$ minimizing this

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