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Neighborhood-restricted [≤2]-achromatic colorings

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ABSTRACT

A (closed) neighborhood-restricted $[\leq 2]$-coloring of a graph G is an assignment of colors to the vertices of G such that no more than two colors are assigned in any closed neighborhood, that is, for every vertex v in G, the vertex v and its neighbors are in at most two different color classes. The $[\leq 2]$-*achromatic number* is defined as the maximum number of colors in any $[\leq 2]$-coloring of G. We study the $[\leq 2]$-achromatic number. In particular, we improve a known upper bound and characterize the extremal graphs for some other known bounds.

1. Introduction

Proper vertex colorings are the most popular type of graph colorings and are well-studied in graph theory. A *proper coloring* of the vertices of a graph G assigns a color to each vertex of G in such a way that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required in any proper coloring of G. In this paper, we study a vertex coloring variation (not necessarily proper) that places a restriction on the maximum number of colors that a vertex and its neighbors may have.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote the *order* of G by $n = n(G) = |V|$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ of vertices adjacent to v, and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* of S is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertices V of a graph G into k color classes V_i. Let $deg_\pi[v] = |\{i : N[v] \cap V_i \neq \emptyset\}|$, that is, $deg_\pi[v]$ equals the number of different colors assigned to vertices in the closed neighborhood $N[v]$ of v by the partition π. For ease of discussion, if the vertices of a set S are assigned colors, then we say that S *contains* these assigned colors.

In [4], a partition π is said to be a $[\leq k]$-*coloring* if $deg_\pi[v] \leq k$ for every vertex $v \in V$, that is, every closed neighborhood contains at most k different colors. The $[\leq k]$-*achromatic number* $\psi_{[\leq k]}(G)$ is the maximum order of a $[\leq k]$-coloring of G, that is, $\psi_{[\leq k]}(G)$ is the maximum number of colors in any $[\leq k]$-coloring of G. If π is a $[\leq k]$-coloring of G with $\psi_{[\leq k]}(G)$ colors,

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then we say that π is a $\psi_{[\leq k]}(G)$-coloring. Note that the trivial partition $\pi = \{V\}$ is a $[\leq k]$-coloring for every integer $k \geq 1$, so $\psi_{[\leq k]}(G) \geq 1$ is defined for all graphs G and all positive integers k.

The $[\leq k]$-colorings were introduced and studied in the literature from a different angle. Such a coloring was called a forbidden rainbow subgraph coloring in [5], a star-$[k]$-coloring in [2], and a 3-consecutive C-coloring in [3]. A subgraph is said to be *rainbow* if under a given coloring, its vertices receive distinct colors. A coloring having no rainbow subgraph F is called a *no-rainbow-F coloring* [5]. The no-rainbow-F coloring, where the subgraph F to be avoided is a star $K_{1,k}$ was studied in [2] and is the same as our $[\leq k]$-coloring. For the special case where $k = 2$, this coloring was defined as a 3-consecutive C-coloring in [3]. A related model for hypergraphs was studied in [7].

As mentioned in [4], neighborhood-restricted colorings are in a sense a dual to the well-known chromatic number. With the chromatic number there is no limit on the number of different colors that can appear in any closed neighborhood, but the objective is to minimize the number of different colors used. With the neighborhood-restricted achromatic number, there is a limit on the number of different colors that can appear in any closed neighborhood, but the objective is to maximize the number of different colors used.

In this paper, we concentrate on the case when $k = 2$, that is, we focus on $[\leq 2]$-colorings and study the $[\leq 2]$-achromatic number. The decision problem associated with calculating $\psi_{[\leq 2]}(G)$ was shown to be NP-hard in [5].

1.1. Terminology

We give some additional notation and terminology that will be used. The 2-*packing number* $\rho(G)$ of a graph G equals the maximum cardinality of a set $S \subset V$ having the property that for any two vertices $u, v \in S$, the distance between u and v is at least three, or equivalently, for any vertex $v \in V$, $|N[v] \cap S| < 2$. A set $S \subseteq V$ is called a *dominating set* if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of any dominating set of G, and a dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$-set. A dominating set S of G is called an *efficient dominating set* if it is also a 2-packing of G. It was shown by Bange et al. in [1] that if a graph G has an efficient dominating set S, then $|S| = \gamma(G)$.

The *corona* $G \circ K_1$ is formed from a graph G by attaching a new vertex v' adjacent to v for each $v \in V(G)$. For any $v \in V$, we denote the graph formed by removing v and all of its incident edges by $G - v$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*.

1.2. Background and aims

The following bounds in terms of diameter are known.

Observation 1 (*[4,5]*). *For any connected graph G with diameter $diam(G)$,*

(i) $\psi_{[\leq 2]}(G) \geq \lceil diam(G)/2 \rceil + 1$, *and*
(ii) $\psi_{[\leq 3]}(G) \geq diam(G) + 1$.

Theorem 2 (*[3]*). *A nontrivial connected graph G has $\psi_{[\leq 2]}(G) = 2$ if and only if $diam(G) \leq 2$.*

In Section 2, we consider the diameter of graphs and determine some Nordhaus–Gaddum type results.

Another lower bound in terms of the 2-packing number is found in [5].

Theorem 3 (*[5]*). *For a graph G, $\psi_{[\leq 2]}(G) \geq \rho(G) + 1$.*

The graphs attaining the bound of Theorem 3 were characterized in [4] as follows.

Theorem 4 (*[4]*). *For any isolate-free graph G, $\psi_{[\leq 2]}(G) \geq \rho(G) + 1$ with equality if and only if G has a $\psi_{[\leq 2]}(G)$-coloring in which at least one color class dominates G.*

The following upper bound on $\psi_{[\leq 2]}(G)$ in terms of the domination number is given in [3].

Theorem 5 (*[3]*). *For any graph G, $\psi_{[\leq 2]}(G) \leq 2\gamma(G)$.*

It is known [6] that the 2-packing number is a lower bound on the domination number of any graph G, that is, $\rho(G) \leq \gamma(G)$. In Section 3, we characterize the graphs attaining the bound of Theorem 5 and improve the bound by showing that, in fact, $\psi_{[\leq 2]}(G) \leq 2\rho(G)$. Hence, we have that $\rho(G) + 1 \leq \psi_{[\leq 2]}(G) \leq 2\rho(G)$. We show every value in this range can be achieved by trees.

An upper bound on $\psi_{[\leq 2]}(G)$ in terms of the order n of a graph G was determined by Goddard, et al. [5].

Theorem 6 (*[5]*). *For a connected graph G of order n, $\psi_{[\leq 2]}(G) \leq \lfloor (n + 2)/2 \rfloor$.*

In Section 4, we give a constructive characterization of the extremal trees for the bound of Theorem 6. Finally, in Section 5, we close with some open problems.