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On the ratio between maximum weight perfect matchings and maximum weight matchings in grids

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ABSTRACT

Given a graph G that admits a perfect matching, we investigate the parameter $\eta(G)$ (originally motivated by computer graphics applications) which is defined as follows. Among all nonnegative edge weight assignments, $\eta(G)$ is the minimum ratio between (i) the maximum weight of a perfect matching and (ii) the maximum weight of a general matching. In this paper, we determine the exact value of η for all rectangular grids, all bipartite cylindrical grids, and all bipartite toroidal grids. We introduce several new techniques to this endeavor.

1. Introduction

All graphs in this paper are finite, undirected and connected. We refer to the textbook of Diestel [3] for any undefined graph terminology. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we define its *neighborhood* as $N(v) = \{u : uv \in E\}$ and the vertices in $N(v)$ are called the *neighbors* of v. The *degree* $d(v)$ of a vertex $v \in V$ is defined as $d(v) = |N(v)|$. The *minimum degree* of G is denoted by $\delta(G)$. The *average degree* of G is defined as follows: $\bar{d}(G) = \frac{1}{n}\sum_{i=1}^{n} d(v_i)$, where $V = \{v_1, \ldots, v_{|V|}\}$. As usual K_n, $n \geq 1$, (resp. $K_{n,m}$, $n, m \geq 1$) denotes the complete graph (resp. complete bipartite graph) on n vertices (resp. with n vertices in one partition and m in the other partition). Finally, C_n, $n \geq 3$, denotes the induced cycle on n vertices.

A *matching* in G is a set $M \subseteq E$ such that no two edges in M share a common vertex. Given a matching M in a graph G, we say that M *saturates* a vertex v and that vertex v is *M-saturated*, if some edge of M is incident to v. A matching M is *perfect* if $|M| = \frac{|V|}{2}$, i.e., all vertices in G are M-saturated. A matching M is *maximal* if there exists no other matching M' such that $M \subseteq M'$ and $|M'| > |M|$. A matching M is *maximum* if it has maximum cardinality.

Let $w : E \to \mathbb{R}^+$ be a *weight function* on the edges of G. We will refer to w as an *edge weighting* of G. Given a subset $E' \subseteq E$, the quantity $w(E') = \sum_{e \in E'} w(e)$ is called the *weight of* E'. A *maximum weight matching* in G, denoted by $M^*(G)$, is a matching of maximum total weight in G. A *maximum weight perfect matching* in G, denoted by $P^*(G)$, is a perfect matching of maximum total weight (among all perfect matchings in G). Given a graph $G = (V, E)$ which admits a perfect matching, the parameter $\eta(G)$ is defined as

$$\eta(G) = \min_{w:E\to\mathbb{R}^+} \frac{w(P^*(G))}{w(M^*(G))}.$$

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The study of the parameter η was initiated in [2] and motivated by applications in computer graphics (see [5,6]) where one seeks to convert a triangle mesh into a quadrangulation. Each triangle is represented by a vertex, two vertices are linked by an edge if the corresponding triangles are adjacent, and edge weights correspond to how "compatible" two triangles are (the definition of compatibility is largely dependent on the specific objective). Due to this application, the first study focused on cubic graphs, i.e. graphs in which all vertices have degree 3. Compared to triangulations, structured grids define a simpler but also widely used mesh [7]. By merging two adjacent grid cells, we obtain an unstructured grid with half as many cells. Thanks to the regular nature of structured grids we can calculate the exact values of η for each grid size, instead of only upper and lower bounds as in the case of cubic graphs.

Since the parameter η can be defined for any graph which admits a perfect matching, its study is of interest from a theoretical point of view. Furthermore, the value of the parameter η tells us how far or close the maximum weight of a perfect matching is from the maximum weight of a general matching in a given graph, considering any nonnegative edge weighting. Thus, it is natural to consider several different graph classes. Notice that the problem of deciding whether $\eta(G) = c$, for a given graph G and a nonnegative real c, is not known to be in P nor to be NP-hard.

It is easy to see that we necessarily have $0 \leq \eta(G) \leq 1$, for any graph G admitting a perfect matching. In [2], the authors characterize those graphs G for which $\eta(G) = 0$ as well as the graphs G for which $\eta(G) = 1$. Furthermore, they provide lower and upper bounds on η for several types of bridgeless cubic graphs, i.e. cubic graphs not containing any edge whose deletion disconnects the graph. Finally, the authors show that if a graph G admits a perfect matching, then the value of $\eta(G)$ is well defined.

The main technique available so far to prove a lower bound on η is the following. Suppose $G = (V, E)$ contains k perfect matchings $P_1, \ldots, P_k$ such that each edge of G belongs to at least r of these matchings. Consider a nonnegative edge weighting w of E. Then $rw(M^*(G)) \leq \sum_{i=1}^{k} w(P_i)$. Without loss of generality, we may assume that $w(P_1) \geq w(P_2) \geq \cdots \geq w(P_k)$. Hence, $rw(M^*(G)) \leq kw(P_1) \leq kw(P^*(G))$ which implies that $\eta(G) \geq \frac{r}{k}$. This proof technique has a major weakness. It cannot prove lower bounds on η higher than 1 over the average degree $\bar{d}(G)$ of G, since $\frac{r}{k}$ is upper bounded by $\frac{1}{\bar{d}(G)}$. Indeed, since the size of a perfect matching is $\frac{|V|}{2}$, it follows from the above that $k\frac{|V|}{2} \geq r|E|$, i.e. $k|V| \geq 2|E|r$. Now using the fact that $2|E| = \sum_{i=1}^{|V|} d(v_i)$, we deduce that $k \geq r\bar{d}(G)$ and so $\frac{r}{k} \leq \frac{1}{\bar{d}(G)}$.

In this paper, we introduce new techniques that break this barrier (see Section 2). These techniques allow us to compute the exact value of η for the following graph classes: (i) rectangular grids; (ii) bipartite cylindrical grids; (iii) bipartite toroidal grids (see Section 3 for the corresponding definitions and theorems). Section 4 is devoted to concluding remarks and open problems.

2. New techniques to determine lower and upper bounds

In this section, we introduce new techniques that enable us to obtain upper and lower bounds on the value of η. We start with a lemma allowing us to determine upper bounds. It generalizes an argument given in [2].

Lemma 1. *Let $G = (V, E)$ be a graph with $\delta(G) \geq 2$. Suppose G contains a perfect matching and there exists a maximal matching that does not saturate a vertex v of degree $\delta(G)$. Then, $\eta(G) \leq \frac{\delta-1}{\delta}$.*

Proof. Consider a vertex v of G with $d(v) = \delta(G)$ and let M be a maximal matching not saturating v. Let $u_1, u_2, \ldots, u_{\delta(G)}$ be the neighbors of v. Since M is maximal, all neighbors of v are necessarily M-saturated. Let u_iw_i, $i = 1, \ldots, \delta(G)$, be the edges of M saturating the neighbors of v. Notice that the neighbors of v may be adjacent, and that the edges of M saturating the neighbors of v are not necessarily distinct. Now any perfect matching P contains at most $\delta(G) - 1$ edges in $\{u_iw_i, i = 1, \ldots, \delta(G)\}$ since P saturates v. Define a nonnegative edge weighting w such that $w(u_iw_i) = 1$ for $i = 1, \ldots, \delta(G)$, and $w(e) = 0$ otherwise. It follows that $w(P) \leq \delta(G) - 1$ and $w(M) = \delta(G)$. Hence, $\eta(G) \leq \frac{\delta(G)-1}{\delta(G)}$. □

The next lemma allows us to obtain lower bounds on η.

Lemma 2. *Let $c, r \geq 0$ be two integers and $G = (V, E)$ be a graph which admits a perfect matching. Let $G_1 = (V, E_1), G_2 = (V, E_2), \ldots, G_k = (V, E_k)$ be k spanning subgraphs of G admitting each a perfect matching and such that $\eta(G_i) \geq c$ for $i = 1, \ldots, k$. If each edge of G is contained in at least r sets among $E_1, E_2, \ldots, E_k$, then $\eta(G) \geq \frac{c\,r}{k}$.*

Proof. First, notice that since $G_1, \ldots, G_k$ are spanning subgraphs of G, any perfect matching in G_i, $i \in \{1, \ldots, k\}$, is also a perfect matching in G. Now, let M be a maximum weight matching of G for some nonnegative edge weighting w. Since each edge of G is contained in at least r sets among $E_1, E_2, \ldots, E_k$, we have

$$\sum_{i=1}^{k} w(M \cap E_i) \geq r\ w(M). \tag{1}$$

Assume, without loss of generality, that $w(M \cap E_1) \geq w(M \cap E_i)$ for $i = 2, \ldots, k$. Then, inequality (1) implies that

$$w(M \cap E_1) \geq \frac{r\ w(M)}{k}. \tag{2}$$