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# Pattern-avoiding alternating words

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ABSTRACT

A word $w = w_1w_2\cdots w_n$ is alternating if either $w_1 < w_2 > w_3 < w_4 > \cdots$ (when the word is up-down) or $w_1 > w_2 < w_3 > w_4 < \cdots$ (when the word is down-up). In this paper, we initiate the study of (pattern-avoiding) alternating words. We enumerate up-down (equivalently, down-up) words via finding a bijection with order ideals of a certain poset. Further, we show that the number of 123-avoiding up-down words of even length is given by the Narayana numbers, which is also the case, shown by us bijectively, with 132-avoiding up-down words of even length. We also give formulas for enumerating all other cases of avoidance of a permutation pattern of length 3 on alternating words.


## 1. Introduction

A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is called *up-down* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5 < \cdots$. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is called *down-up* if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \pi_5 > \cdots$. A famous result of André is saying that if $E_n$ is the number of up-down (equivalently, down-up) permutations of $1, 2, \ldots, n$, then

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Some aspects of up-down and down-up permutations, also called *reverse alternating* and *alternating*, respectively, are surveyed in [15]. Slightly abusing these definitions, we refer to *alternating permutations* as the union of up-down and down-up permutations. This union is known as the set of *zigzag permutations*.

The study of alternating permutations was extended to other types of alternating sequences defined in various ways, for example, those related to words [3,9,13] and to compositions [2,5]. In this paper, we study *alternating words*. These words, also called *zigzag words*, are the union of up-down and down-up words, which are defined in a similar way to the definition of up-down and down-up permutations, respectively. For example, 1214, 2413, 2424 and 3434 are examples of up-down words of length 4 over the alphabet $\{1, 2, 3, 4\}$.

Section 2 is dedicated to the enumeration of up-down words, which is equivalent to enumerating down-up words by applying the operation of *complement*. For a word $w = w_1w_2\cdots w_n$ over the alphabet $\{1, 2, \ldots, k\}$ its complement $w^c$

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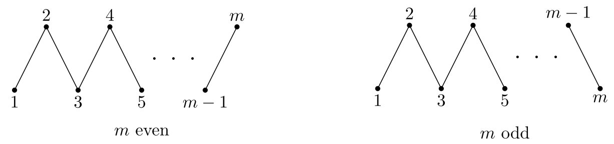


**Fig. 1.** The zigzag poset $Z_m$.

**Table 1**
The number $N_{k,\ell}$ of down-up words on $[k]$ of length $\ell$ for small values of $k$ and $\ell$.

| $k$ | $\ell$ | | | | | | | | | | | OEIS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | Trivial |
| 3 | 1 | 3 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | A000045 |
| 4 | 1 | 4 | 6 | 14 | 31 | 70 | 157 | 353 | 793 | 1 782 | 4 004 | A006356 |
| 5 | 1 | 5 | 10 | 30 | 85 | 246 | 707 | 2 037 | 5 864 | 16 886 | 48 620 | A006357 |
| 6 | 1 | 6 | 15 | 55 | 190 | 671 | 2353 | 8 272 | 29 056 | 102 091 | 358 671 | A006358 |
| 7 | 1 | 7 | 21 | 91 | 371 | 1547 | 6405 | 26 585 | 110 254 | 457 379 | 1897 214 | A006359 |

is the word $c_1c_2\cdots c_n$, where for each $i = 1, 2, \dots, n$, $c_i = k + 1 - w_i$. For example, the complement of the word 24265 over the alphabet $\{1, 2, \dots, 6\}$ is 53512. Our enumeration in Section 2 is done by linking bijectively up-down words to order ideals of certain posets and using known results. We note that an alternative enumeration of these words (in terms of generating functions) is done in [3] (see formula (1.11) there). However, as far as we can see, our recursive formula (1) allowing quick computation of numbers in question cannot be easily derived from the generating functions in [3].

A (*permutation*) *pattern* is a permutation $\tau = \tau_1\tau_2\cdots\tau_k$. We say that a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ *contains an occurrence* of $\tau$ if there are $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to $\tau$. If $\pi$ does not contain an occurrence of $\tau$, we say that $\pi$ *avoids* $\tau$. For example, the permutation 315267 contains several occurrences of the pattern 123, for example, the subsequences 356 and 157, while this permutation avoids the pattern 321. Occurrences of a pattern in words are defined similarly as subsequences order-isomorphic to a given word called pattern (the only difference with permutation patterns is that word patterns can contain repetitive letters, which is not in the scope of this paper).

Comprehensive introductions to the theory of patterns in permutations, words and compositions can be found in [7,8]. In particular, Section 6.1.8 in [8] discusses known results on pattern-avoiding alternating permutations (also, see [4,10]), and Section 7.1.6 discusses results on permutations avoiding patterns in a more general sense. An example of relevant studies is that of longest alternating subsequences in pattern-avoiding words conducted in [11].

In this paper we initiate the study of pattern-avoiding alternating words. In Section 3 we enumerate up-down words over $k$-letter alphabet avoiding the pattern 123. In particular, we show that in the case of even length, the answer is given by the *Narayana numbers* counting, for example, *Dyck paths* with a specified number of peaks (see Theorem 3.2). Interestingly, the number of 132-avoiding words over $k$-letter alphabet of even length is also given by the Narayana numbers, which we establish bijectively in Section 4. In Section 5 we provide a (non-closed form) formula for the number of 132-avoiding words over $k$-letter alphabet of odd length. In Section 6 we show that the enumeration of 312-avoiding up-down words is equivalent to that of 123-avoiding up-down words. Further, a classification of all cases of avoiding a length 3 permutation pattern on up-down words is discussed in Section 7. Finally, some concluding remarks are given in Section 8.

In what follows, $[k] = \{1, 2, \dots, k\}$.

## 2. Enumeration of up-down words

In this section, we consider the enumeration of up-down words. We shall show that this problem is the same as that of enumerating order ideals of a certain poset. Since up-down words are in one-to-one-correspondence with down-up words by using the complement operation, we consider only down-up words throughout this section.

Table 1 provides the number $N_{k,\ell}$ of down-up words of length $\ell$ over the alphabet $[k]$ for small values of $k$ and $\ell$ indicating connections to the *Online Encyclopedia of Integer Sequences* (*OEIS*) [14].

We assume the reader is familiar with the notion of a partially ordered set (poset) and some basic properties of posets; e.g. see [16]. A partially ordered set $P$ is a set together with a binary relation denoted by $\le_P$ that satisfies the properties of reflexivity, antisymmetry and transitivity. An order ideal of $P$ is a subset $I$ of $P$ such that if $x \in I$ and $y \le x$ then $y \in I$. We denote $J(P)$ the set of all order ideals of $P$.

Let $\mathbf{n}$ be the poset on $[n]$ with its usual order ($\mathbf{n}$ is a linearly ordered set). The *m-element zigzag poset*, denoted $Z_m$, is shown schematically in Fig. 1. Note that the order $<_{Z_m}$ in $Z_m$ is $1 < 2 > 3 < 4 > 5 < \cdots$. The definition of the order $\le_{Z_m}$ is self-explanatory.

The poset $Z_m \times \mathbf{n}$ is as shown in Fig. 2. Elements of $Z_m \times \mathbf{n}$ are pairs $(i, j)$, where $i \in Z_m$ and $j \in [n]$, and the order is defined as follows:

$$(i, j) \le (k, \ell) \quad \text{if and only if } i \le_{Z_m} k \text{ and } j \le \ell.$$