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The competition graphs of oriented complete bipartite graphs

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ABSTRACT

In this paper, we study the competition graphs of oriented complete bipartite graphs. We characterize graphs that can be represented as the competition graphs of oriented complete bipartite graphs. We also present the graphs having the maximum number of edges and the graphs having the minimum number of edges among such graphs.

1. Introduction

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph G defined by $V(G) = V(D)$ and $E(G) = \{uv \mid u, v \in V(D), u \neq v, N_D^+(u) \cap N_D^+(v) \neq \emptyset\}$, where $N_D^+(x)$ denotes the set of out-neighbors of a vertex x in D. We denote the set of in-neighbors of a vertex x in a digraph D by $N_D^-(x)$ and denote the set of neighbors of a vertex x in a graph G by $N_G(x)$. Competition graphs arose in connection with an application in ecology (see [2]) and also have applications in coding, radio transmission, and modeling of complex economic systems. Early literature of the study on competition graphs is summarized in the survey papers by Kim [6] and Lundgren [10].

For a digraph D, the *underlying graph* of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{uv \mid (u, v) \in A(D)\}$. An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G. An *oriented graph* is a graph with an orientation. A *tournament* is an oriented complete graph. The competition graphs of tournaments have been actively studied (see [1,3,5], and [4] for papers related to this topic).

It seems to be a natural shift to take a look at the competition graphs of orientations of complete bipartite graphs. First, we can observe that the competition graph of an orientation of a complete bipartite graph is a disconnected graph as follows.

Proposition 1.1. *Let D be an orientation of a complete bipartite graph $K_{m,n}$ with bipartition (U, V), where $|U| = m$ and $|V| = n$. Then, the competition graph of D has no edges between the vertices in U and the vertices in V.*

Proof. Since D is an orientation of $K_{m,n}$, $N_D^+(x) \cup N_D^-(x) = N_{K_{m,n}}(x)$ holds for any vertex x in D. Take any vertex u in U and any vertex v in V. Since $N_{K_{m,n}}(u) \subseteq V$, $N_{K_{m,n}}(v) \subseteq U$, and $U \cap V = \emptyset$, we have $N_{K_{m,n}}(u) \cap N_{K_{m,n}}(v) = \emptyset$. Therefore, $N_D^+(u) \cap N_D^+(v) = \emptyset$. Thus, there is no edge between u and v in the competition graph of D. Hence the proposition holds. $\square$

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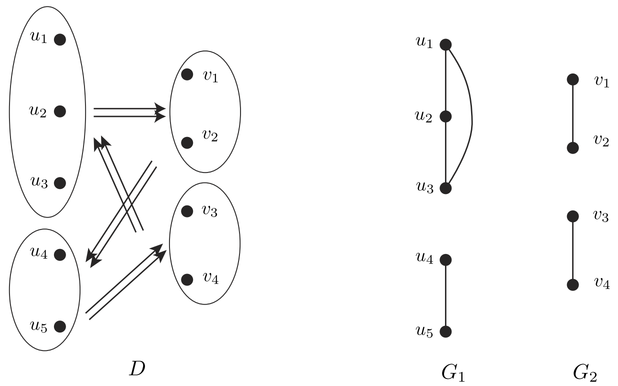

Fig. 1. An orientation D of the complete bipartite graph $K_{5,4}$ and its competition graph $G_1 \cup G_2$. (A pair of parallel arrows means that there is an arc from each vertex in the ellipse from which the arc initiates to each vertex in the ellipse to which the arc terminates.)

Based on Proposition 1.1, we introduce the notion of competition-realizable pairs.

Definition 1. Let G_1 and G_2 be graphs with m vertices and n vertices, respectively. The pair (G_1, G_2) is said to be *competition-realizable through* $K_{m,n}$ (in this paper, we only consider orientations of $K_{m,n}$ and therefore we omit "through $K_{m,n}$") if the disjoint union of G_1 and G_2 is the competition graph of an orientation of the complete bipartite graph $K_{m,n}$ with bipartition $(V(G_1), V(G_2))$.

Let us see an example. Let G_1 be the graph defined by $V(G_1) = \{u_1, u_2, u_3, u_4, u_5\}$ and $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3, u_4u_5\}$, and let G_2 be the graph defined by $V(G_2) = \{v_1, v_2, v_3, v_4\}$ and $E(G_2) = \{v_1v_2, v_3v_4\}$. Then the pair $(G_1, G_2) \cong (K_3 \cup K_2, K_2 \cup K_2)$ is competition-realizable through $K_{5,4}$ (see Fig. 1).

In this paper, we study the competition graphs of oriented complete bipartite graphs by using the notion of competition-realizable pairs. We characterize graphs that can be represented as the competition graphs of oriented complete bipartite graphs. We also present the graphs having the maximum number of edges and the graphs having the minimum number of edges among such graphs.

2. A characterization of competition-realizable pairs in terms of edge clique covers

In this section, we present a theorem which characterizes a competition-realizable pair (G_1, G_2) in terms of edge clique covers of the graphs G_1 and G_2 without mentioning oriented complete bipartite graphs.

By a *family*, we mean a multiset of subsets of a set. A *clique* of a graph G is a set of vertices of G in which any two vertices are adjacent in G. We also consider an empty set $\emptyset$ as a clique. An *edge clique cover* of a graph G is a family $\mathcal{F}$ of cliques of G such that, for any two adjacent vertices of G, there is a clique in $\mathcal{F}$ containing both of them. For a graph G, we denote by $\theta_E(G)$ the minimum size of an edge clique cover of G.

The *intersection graph* $\Omega(\mathcal{F})$ of a family $\mathcal{F}$ is the graph whose vertex set is $\mathcal{F}$ and in which two sets X and Y in $\mathcal{F}$ are adjacent if and only if $X \cap Y \neq \emptyset$. Recall that a *graph isomorphism* from G_1 to G_2 is a bijection φ from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $\varphi(x)\varphi(y) \in E(G_2)$. For a family $\mathcal{F}$ of subsets of a set V, the *dual family* of $\mathcal{F}$ is the family $\mathcal{F}^*$ defined by $\mathcal{F}^* = \{V \setminus S \mid S \in \mathcal{F}\}$.

Theorem 2.1. *Let G_1 and G_2 be graphs. Then, (G_1, G_2) is a competition-realizable pair if and only if there exist edge clique covers $\mathcal{F}_1$ and $\mathcal{F}_2$ of G_1 and G_2, respectively, such that*

(i) *there exist graph isomorphisms $\varphi_1 : G_2 \to \Omega(\mathcal{F}_1^*)$ and $\varphi_2 : G_1 \to \Omega(\mathcal{F}_2^*)$, where $\mathcal{F}_i^* := \{V(G_i) \setminus S \mid S \in \mathcal{F}_i\}$ for $i = 1, 2$;*
(ii) *for $u \in V(G_1)$ and $v \in V(G_2)$, $u \in \varphi_1(v)$ if and only if $v \notin \varphi_2(u)$.*

Before proving the theorem, let us see an example. Let

$$\mathcal{F}_1 = \{\{u_1, u_2, u_3\}, \{u_1, u_2, u_3\}, \{u_4, u_5\}, \{u_4, u_5\}\},$$
$$\mathcal{F}_2 = \{\{v_1, v_2\}, \{v_1, v_2\}, \{v_1, v_2\}, \{v_3, v_4\}, \{v_3, v_4\}\}.$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are edge clique covers of $G_1 := K_3 \cup K_2$ and $G_2 := K_2 \cup K_2$, respectively, and we have $\mathcal{F}_1^* = \mathcal{F}_1$ and $\mathcal{F}_2^* = (\mathcal{F}_2 - \{\{v_1, v_2\}\}) \cup \{\{v_3, v_4\}\}$. We define maps φ_1 from $V(G_2)$ to $\mathcal{F}_1^*$ and φ_2 from $V(G_1)$ to $\mathcal{F}_2^*$ by $\varphi_1(v_1) = \varphi_1(v_2) = \{u_4, u_5\}$; $\varphi_1(v_3) = \varphi_1(v_4) = \{u_1, u_2, u_3\}$; $\varphi_2(u_1) = \varphi_2(u_2) = \varphi_2(u_3) = \{v_1, v_2\}$; $\varphi_2(u_4) = \varphi_2(u_5) = \{v_3, v_4\}$. It is easy to check that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy conditions (i) and (ii) of Theorem 2.1.

Lemma 2.2. *Let G_1 and G_2 be graphs. Let D be an orientation of a complete bipartite graph with bipartition $(V(G_1), V(G_2))$ such that the competition graph of D is $G_1 \cup G_2$. Then, the family $\{N_D^-(v) \mid v \in V(G_2)\}$ is an edge clique cover of G_1, and the family $\{N_D^-(u) \mid u \in V(G_1)\}$ is an edge clique cover of G_2.*