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Note

On domination number and distance in graphs

Cong X. Kang

Texas A&M University at Galveston, Galveston, TX 77553, USA

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ABSTRACT

A vertex set S of a graph G is a *dominating set* if each vertex of G either belongs to S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of S as S varies over all dominating sets of G. It is known that $\gamma(G) \geq \frac{1}{3}(diam(G)+1)$, where $diam(G)$ denotes the diameter of G. Define C_r as the largest constant such that $\gamma(G) \geq C_r \sum_{1\leq i<j\leq r} d(x_i, x_j)$ for any r vertices of an arbitrary connected graph G; then $C_2 = \frac{1}{3}$ in this view. The main result of this paper is that $C_r = \frac{1}{r(r-1)}$ for $r \geq 3$. It immediately follows that $\gamma(G) \geq \mu(G) = \frac{1}{n(n-1)} W(G)$, where $\mu(G)$ and $W(G)$ are respectively the average distance and the Wiener index of G of order n. As an application of our main result, we prove a conjecture of DeLaViña et al. that $\gamma(G) \geq \frac{1}{2}(ecc_G(B)+1)$, where $ecc_G(B)$ denotes the eccentricity of the boundary of an arbitrary connected graph G.

1. Introduction

We consider finite, simple, undirected, and connected graphs $G = (V(G), E(G))$ of order $|V(G)| \geq 2$ and size $|E(G)|$. For $W \subseteq V(G)$, we denote by $\langle W\rangle_G$ the subgraph of G induced by W. For $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. Further, let $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$ for $S \subseteq V(G)$. The degree of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$. The *distance* between two vertices $x, y \in V(G)$ in the subgraph H, denoted by $d_H(x, y)$, is the length of a shortest path between x and y in the subgraph H. The *diameter* $diam(H)$ of a graph H is $\max\{d_H(x, y) \mid x, y \in V(H)\}$.

A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The *domination number* (resp. *total domination number*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of S as S varies over all dominating sets (resp. total dominating sets) in G; a dominating set (resp. total dominating set) of G of minimum cardinality is called a $\gamma(G)$-set (resp. $\gamma_t(G)$-set).

Both distance and (total) domination are very well-studied concepts in graph theory. For a survey of the myriad variations on the notion of domination in graphs, see [4].

It is well-known that $\gamma(G) \geq \frac{1}{3}(diam(G)+1)$ $(*)$; a "proof" to $(*)$ can be found on p. 56 of the authoritative reference [4]. However, the "proof" contained therein is logically flawed. We provide a counter-example to a crucial assertion in the "proof", and then present a correct proof to $(*)$. Upon some reflection, we see that $(*)$ is the two parameter case of a family of inequalities existing between $\gamma(G)$ and the distances in G, in the following way: $\gamma(G) \geq \frac{1}{3}(diam(G)+1) = \frac{1}{3\binom{r}{2}}\left(\binom{r}{2} diam(G) + \binom{r}{2}\right) \geq \frac{1}{3\binom{r}{2}}\left(\sum_{1\leq i<j\leq r} d(x_i, x_j)\right)$. The inequality $\gamma(G) \geq \frac{1}{3\binom{r}{2}}\left(\sum_{1\leq i<j\leq r} d(x_i, x_j)\right)$ naturally brings up the question: what is the largest constant C_r, such that $\gamma(G) \geq C_r\left(\sum_{1\leq i<j\leq r} d(x_i, x_j)\right)$, for all connected graphs $G = (V, E)$ and arbitrary vertices $x_1, \ldots, x_r \in V$, where $r \geq 2$? Taking this viewpoint, we have $C_2 = \frac{1}{3}$ by $(*)$.

E-mail address: kangc@tamug.edu.

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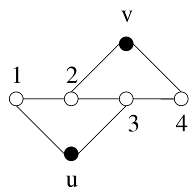

Fig. 1. A counter-example.

The main result of this paper is that $C_r = \frac{1}{r(r-1)}$ for $r \geq 3$. Since, for a graph G of order n, $W(G) = \sum_{1\leq i<j\leq n} d(x_i, x_j)$ is the *Wiener index* of G (see [6]) and $\mu(G) = \frac{1}{n(n-1)}W(G)$ is the average distance (per definition found in [1]), it follows that $\gamma(G) \geq \mu(G) = \frac{1}{n(n-1)}W(G)$. As an application of our main result, we prove a conjecture in [3] by DeLaViña et al. that $\gamma(G) \geq \frac{1}{2}(ecc_G(B) + 1)$, where $ecc_G(B)$ denotes the eccentricity of the boundary of an arbitrary connected graph G (to be defined in Section 4).

This paper is motivated by the work of Henning and Yeo in [5], where they obtained similar inequalities for total domination number γ_t (rather than domination number γ). Given the close relation between the two graph parameters, we expect the techniques used in [5] to be readily adaptable towards the results of this paper. However, in striking contrast to [5], we avoid the painstaking case-by-case, structural analysis employed there by making use of the easy and well-known Lemma 3.1; this results in a much simpler and shorter paper. Further, we are able to obtain (in domination) the exact value of C_r for every r, rather than only a bound (in total domination, c.f. [5]) for C_r for all but the first few values of r.

2. An Error in the proof of $\gamma(G) \geq \frac{1}{3}(diam(G) + 1)$ in FoDiG

For readers' convenience, we first reproduce Theorem 2.24 and its incorrect proof as it appears on p. 56 of [4], the authoritative reference in the field of domination titled *Fundamentals of Domination in Graphs*.

Theorem 2.1. *For any connected graph G,* $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma(G)$.

"Proof", (as found on p. 56 of [4]). Let S be a γ-set of a connected graph G. Consider an arbitrary path of length $diam(G)$. This diametral path includes at most two edges from the induced subgraph $\langle N[v]\rangle$ for each $v \in S$. Furthermore, since S is a γ-set, the diametral path includes at most $\gamma(G) - 1$ edges joining the neighborhoods of the vertices of S. Hence, $diam(G) \leq 2\gamma(G) + \gamma(G) - 1 = 3\gamma(G) - 1$ and the desired result follows. □

Presumably, by a "diametral path", the authors had in mind an induced path with length $diam(G)$. Still, the assertion of the sentence beginning with "Furthermore" is incorrect, as seen by the example in Fig. 1: notice that $S = \{u, v\}$ is a γ-set and the vertices 1, 2, 3, 4 form a diametral path containing 3 edges joining $\langle N[u]\rangle$ with $\langle N[v]\rangle$, whereas $\gamma(G) - 1 = 1$.

3. Domination number and distance in graphs

The following lemma can be proved by exactly the same argument given in the proof of Lemma 2 in [2]; it was also observed on p. 23 of [1].

Lemma 3.1 ([1,2]). *Let M be a $\gamma(G)$-set. Then there is a spanning tree T of G such that M is a $\gamma(T)$-set.*

Now, we apply Lemma 3.1 to give a correct proof of Theorem 2.1.

Proof of Theorem 2.1. Given G, take a spanning tree T of G such that $\gamma(G) = \gamma(T)$. Suppose, for the sake of contradiction, $\gamma(G) < \frac{1}{3}(diam(G) + 1)$. Since $\gamma(T) = \gamma(G)$ and $diam(T) \geq diam(G)$, we have

$$\gamma(T) < \frac{1}{3}(diam(T) + 1). \tag{1}$$

Take a path P of T with length equal to $diam(T)$. If (1) holds, there must exist a vertex u of T such that $|V(P) \cap N[u]| \geq 4$. Since P is a path of T (a tree), this is impossible. □

Theorem 3.2. *Given any three vertices x_1, x_2, x_3 of a connected graph G, we have*

$$\gamma(G) \geq \frac{1}{6}(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)). \tag{2}$$

Further, if equality is attained in (2), then $d_G(u, v) \equiv 2 \pmod 3$ for any pair $u, v \in \{x_1, x_2, x_3\}$.