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Independent domination in finitely defined classes of graphs: Polynomial algorithms

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ABSTRACT

We study the problem of finding in a graph an inclusionwise maximal independent set of minimum cardinality, known as MINIMUM MAXIMAL INDEPENDENT SET or INDEPENDENT DOMINATING SET problem. This is one of the hardest problems in algorithmic graph theory. In particular, restricted to the class of so called SAT-graphs, this problem coincides with SATISFIABILITY, the central problem of theoretical computer science. The class of SAT-graphs, and many other important graph classes, such as graphs of bounded vertex degree or line graphs, can be characterized by finitely many forbidden induced subgraphs. We call such classes finitely defined. The paper [R. Boliac and V. Lozin, Independent domination in finitely defined classes of graphs, Theoretical Computer Science, 301 (2003) 271–284] reveals various conditions under which the INDEPENDENT DOMINATING SET problem remains NP-hard in a finitely defined class. In the present paper, we identify a number of finitely defined classes where the problem admits polynomial-time solutions. In particular, we prove that the problem is solvable in polynomial time in the class of $P_2 + P_3$-free graphs by correcting a mistake of the first two authors of the paper in their earlier solution. This result is in a sharp contrast with the fact that in the class of $P_3 + P_3$-free graphs the problem is known to be NP-hard, since this class contains all SAT-graphs.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. An *independent set* in G is a subset of vertices no two of which are adjacent. A *dominating set* in G is a subset $U \subseteq V$ such that every vertex in $V \setminus U$ has a neighbour in U. The INDEPENDENT DOMINATING SET problem (or simply INDEPENDENT DOMINATION) is the problem of finding in a graph an independent dominating set of minimum cardinality. Clearly, an independent set is dominating if and only if it is *maximal*, i.e. not contained in any larger independent set. That is why INDEPENDENT DOMINATION is also known as MINIMUM MAXIMAL INDEPENDENT SET.

Computationally, this is a difficult problem, i.e. it is NP-hard. Moreover, the problem remains NP-hard under substantial restrictions, for instance, for graphs of bounded vertex degree, line graphs [26], bipartite graphs [8], etc. Of particular interest in this list of classes where the problem is NP-hard is the class of so called SAT-graphs. When restricted to the class of SAT-graphs the problem becomes equivalent to SATISFIABILITY, the central problem of theoretical computer science. In other words, INDEPENDENT DOMINATION in the class of SAT-graphs can be viewed as SATISFIABILITY described in graph-theoretic terms. We discuss the relationship between the two problems in more detail in Section 2.

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An important property of the class of SAT-graphs is that it can be characterized by *finitely many* forbidden induced subgraphs. We call classes of graphs possessing this property *finitely defined*. The family of finitely defined classes contains many classes of theoretical or practical importance, such as graphs of degree at most k (for a fixed k), line graphs [16], triangle-free graphs, cographs, split graphs etc. In some classes of this family the INDEPENDENT DOMINATING SET problem is NP-hard (which is the case, for instance, for graphs of degree at most $k \geq 3$, line graphs, triangle-free graphs), while in some others (such as cographs or split graphs) the problem can be solved in polynomial time. Various conditions under which the problem remains NP-hard in a finitely defined class have been revealed in [3]. We survey these conditions in Section 2, where the reader can also find all preliminary information related to the topic of the paper.

In the present paper, we look at the problem from the polynomial side, i.e. we study finitely defined classes for which the NP-hardness conditions revealed in [3] fail and derive for some of these classes polynomial-time algorithms. Some of our results deal with a more general version of the problem in which each vertex is assigned a positive integer, the *weight* of the vertex, and the problem consists in finding an independent dominating set of minimum total weight. We call it WEIGHTED INDEPENDENT DOMINATING SET problem or simply WEIGHTED INDEPENDENT DOMINATION. Let us observe that the complexity of the two versions of the problem may differ for graphs in the same class. For instance, INDEPENDENT DOMINATION for chordal graphs can be solved in polynomial time [11], while WEIGHTED INDEPENDENT DOMINATION in this class is NP-hard [6].

2. Preliminaries

All graphs in this paper are finite, undirected, without loops and multiple edges. Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. For a subset $U \subseteq V(G)$, we denote by $N_G(U)$ the *neighbourhood* of U in G, i.e. the subset of vertices of G outside U each of which has a neighbour in U, and we denote by $A_G(U)$ the *antineighbourhood* of U in G, i.e. the subset of vertices of G outside U none of which has a neighbour in U. If the graph G is clear from the context, we omit the subscript and write $N(U)$ and $A(U)$. Also, if U consists of a single vertex, say $U = \{v\}$, we omit curly brackets and write $N(v)$ and $A(v)$. The degree of a vertex v is the number of its neighbours, i.e. $|N(v)|$.

As usual, P_n, C_n and K_n denote a chordless path, a chordless cycle and a complete graph on n vertices, respectively. Also, $K_{n,m}$ is a complete bipartite graph with parts of size n and m. Given a graph G, we denote by mG the disjoint union of m copies of G.

The subgraph of G induced by a subset $U \subseteq V(G)$ is denoted $G[U]$. If a graph H is isomorphic to an induced subgraph of G we say that G contains H. Otherwise, we say that G is H-free and call H a *forbidden induced subgraph* for G.

Many important graph classes can be described by means of forbidden induced subgraphs. For instance, bipartite graphs (i.e. graphs whose vertices can be partitioned into at most two independent sets) are precisely graphs containing no odd cycles, and split graphs (i.e. graphs whose vertices can be partitioned into an independent set and a clique) are precisely $(2K_2, C_4, C_5)$-free graphs.

It is known that in the class of bipartite graphs INDEPENDENT DOMINATION is NP-hard, while for split graphs the problem can be solved in polynomial time. Let us observe that split graphs can also be defined as graphs whose vertices can be partitioned into a clique and a graph of degree 0. Now let us consider an extension of this class to the class $\mathcal{P}$ of graphs whose vertices can be partitioned into a clique and a graph of degree at most one. The class $\mathcal{P}$ is of particular interest for the INDEPENDENT DOMINATING SET problem because it contains so-called SAT-graphs.

A SAT-graph is a graph G representing an instance of the SATISFIABILITY problem [27]. The vertices in the clique part of G represent clauses and the vertices in the other part represent literals (i.e. variables and their negations). Each literal vertex x is connected to its negation $\bar{x}$ and to the clauses containing it. It is not difficult to see that every independent dominating set I in G contains *exactly* one vertex in each pair x, $\bar{x}$. If I dominates (satisfies) each clause, the formula is satisfiable. Determining if G contains an independent dominating set satisfying all clauses is equivalent to (coincides with) SATISFIABILITY. Therefore, INDEPENDENT DOMINATION is NP-hard for graphs in the class $\mathcal{P}$.

Similarly to split graphs, the class $\mathcal{P}$ can be characterized by *finitely many* forbidden induced subgraphs [27]. As we mentioned in the introduction, we call classes of graphs possessing this property *finitely defined*. Several sufficient conditions for the NP-hardness of INDEPENDENT DOMINATION in finitely defined classes have been identified in [3]. To describe some of them, let us denote by $\mathcal{S}$ the class of forests of degree at most 3, in which every connected component contains at most one vertex of degree 3. Also, by $\mathcal{T}$ we denote the class of line graphs of graphs in $\mathcal{S}$. In other words, $\mathcal{S}$ is the class of graphs in which every connected component has the form $S_{i,j,k}$ represented in Fig. 1(left) and $\mathcal{T}$ is the class of graphs in which every connected component has the form $T_{i,j,k}$ represented in Fig. 1(right). Observe that in both cases, each of the indices i, j, k can be equal to 0.

Theorem 1 *([3]). Let M be a finite set. If $M \cap \mathcal{S} = \emptyset$ or $M \cap \mathcal{T} = \emptyset$, then the* INDEPENDENT DOMINATING SET *problem is NP-hard in the class of M-free graphs.*

In the family of finitely defined classes, of particular interest are *monogenic* classes, i.e. classes defined by a single forbidden induced subgraph H. Under the assumption that $P \neq NP$, Theorem 1 implies that INDEPENDENT DOMINATION is polynomial-time solvable in the class of H-free graphs *only if* $H \in \mathcal{S} \cap \mathcal{T}$. It is not difficult to see that the intersection $\mathcal{S} \cap \mathcal{T}$ consists of linear forests, i.e. graphs every connected component of which is a path. Let us denote this class by $\mathcal{L}$ and let us discuss the complexity of the problem in H-free graphs for some small graphs $H \in \mathcal{L}$.