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The complexity of forbidden subgraph sandwich problems and the skew partition sandwich problem☆

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ABSTRACT

The Π graph sandwich problem asks, for a pair of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$, whether there exists a graph $G = (V, E)$ that satisfies property Π and $E_1 \subseteq E \subseteq E_2$. We consider the property of being F-free, where F is a fixed graph. We show that the claw-free graph sandwich and the bull-free graph sandwich problems are both NP-complete, but the paw-free graph sandwich problem is polynomial. This completes the study of all cases where F has at most four vertices. A skew partition of a graph G is a partition of its vertex set into four nonempty parts A, B, C, D such that each vertex of A is adjacent to each vertex of B, and each vertex of C is nonadjacent to each vertex of D. We prove that the skew partition sandwich problem is NP-complete, establishing a computational complexity non-monotonicity.

1. Introduction

All graphs considered here are finite and undirected. Given a graph property Π, the Π GRAPH SANDWICH PROBLEM is defined as follows:

Input: A pair (G_1, G_2) of graphs with $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $E_1 \subseteq E_2$;

Question: Is there a graph $G = (V, E)$ that satisfies property Π and inclusion $E_1 \subseteq E \subseteq E_2$?

The graph sandwich problem was introduced by Golumbic and Shamir in [16] and further studied in [13,17]. Clearly, when $G_1 = G_2$ the problem is to decide whether G_1 satisfies property Π. So the graph sandwich problem generalizes the recognition problem of deciding whether a graph satisfies a given property. In particular, if the recognition problem is NP-complete, then the sandwich problem is also NP-complete. When the property Π is to belong to a class $\mathcal{C}$ of graphs, we may also speak of the $\mathcal{C}$ *graph sandwich problem*. Golumbic, Kaplan and Shamir [14,15] proved that the interval graph, unit interval graph, permutation graph and comparability graph sandwich problems are all NP-complete, while the split graph, threshold graph and cograph sandwich problems are in P. Graph sandwich problems have attracted much attention

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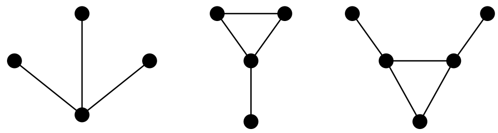

Fig. 1. Claw, paw and bull.

because of many applications and as a natural generalization of recognition problems. See e.g. [1,7,6,8,10,18,19,25] where many results on sandwich problems for graph properties and classes were obtained.

We say that a graph G *contains* a graph F if some induced subgraph of G is isomorphic to F. A graph G is F*-free* if it does not contain F. Dantas, de Figueiredo, da Silva and Teixeira [6] initiated a study of the F-free graph sandwich problem, and determined the complexity status (in P or NP-complete) of the problem for several graphs F, including the cases when F is the diamond ($K_4 \setminus e$) and when F is the 4-hole C_4. Here, we develop this study by determining the complexity status for the cases where F has four vertices that were not solved in [6]. We also consider some graphs on five vertices or more.

A long-standing open problem posed in the seminal paper by Golumbic, Kaplan and Shamir [15] is the complexity of the perfect graph sandwich problem. The celebrated proof of the Strong Perfect Graph Theorem by Chudnovsky et al. [3] established the characterization of perfect graphs by forbidden induced subgraphs; in that proof, the concept of skew partition plays a key role. A *skew partition* of a graph $G = (V, E)$ is a partition of its vertex set V into four nonempty parts A, B, C, D such that each vertex of part A is adjacent to each vertex of part B, and each vertex of part C is nonadjacent to each vertex of part D. Note that if (A, B, C, D) is a skew partition of G then (C, D, A, B) is a skew partition of $\overline{G}$ and vice versa. It is this self-complementarity which first suggested that these partitions might be important to an understanding of the structure of perfect graphs. Chvátal [4] introduced the concept of skew partition. He conjectured that no minimal imperfect graph admits a skew partition and speculated that skew partitions might play a key role in a decomposition theorem which would imply the Strong Perfect Graph Conjecture. Chvátal [4] defined a *star cutset* as any cutset that contains a vertex adjacent to every other vertex of the cutset. If G contains a skew partition (A, B, C, D), then $A \cup B$ is a *skew cutset* of G separating $G[C]$ from $G[D]$. If A is a clique of G, then G has a star cutset. If both A and B are cliques of G, then G has a clique cutset. Chvátal [4] proved that no minimal imperfect graph has a star cutset and conjectured that no minimal imperfect graph has a skew cutset. Polynomial-time algorithms for the recognition of clique cutset [28], star cutset [4], and skew cutset [9] were established. Recent work has further studied the computational complexity of skew cutsets [20,22,27].

Regarding the corresponding sandwich problems CLIQUE CUTSET SANDWICH PROBLEM is NP-complete, but STAR CUTSET SANDWICH PROBLEM is polynomial [26]. We prove that the further generalization SKEW PARTITION SANDWICH PROBLEM is NP-complete establishing an interesting computational complexity non-monotonicity. Note that homogeneous set and clique cutset are vertex partitions into three nonempty parts. The dichotomy polynomial time versus NP-complete was completely determined for the class of three nonempty part sandwich problems [24].

Let us recall some basic definitions and notation. For any integer $k \geq 4$, a k*-hole* is a chordless cycle of length k and is denoted by C_k. For any integer $k \geq 1$, we let K_k and P_k respectively denote the complete graph and the path on k vertices. Let $K_p \setminus e$ denote the complete graph on p vertices minus one edge. For any graph G and subset X of vertices of G, we let $G[X]$ denote the subgraph of G induced by X. The complementary graph of G is denoted by $\overline{G}$. Given two vertex-disjoint graphs F_1 and F_2, their *union* is the graph $F_1 + F_2$ with vertex-set $V(F_1) \cup V(F_2)$ and edge-set $E(F_1) \cup E(F_2)$. The union of k copies of one graph G is denoted by kG.

For an instance (G_1, G_2) of the sandwich problem with $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $E_1 \subseteq E_2$, we say that any member of E_1 is a *forced edge*, any member of $E_{opt} = E_2 \setminus E_1$ is an *optional edge*, and any pair in $V \times V \setminus E_2$ is a *forbidden pair*. Every graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$ is called a sandwich graph for the pair (G_1, G_2). In that case, E consists of all forced edges plus some (possibly zero) optional edges, and E contains no edge corresponding to a forbidden pair.

We recall some results from [6,15].

- *Complementary graphs* [15]. For any fixed graph F, if G is an F-free sandwich graph for the pair (G_1, G_2), then $\overline{G}$ is an $\overline{F}$-free sandwich graph for the pair $(\overline{G_2}, \overline{G_1})$. It follows that the F-free graph sandwich problem and the $\overline{F}$-free graph sandwich problem have the same complexity.
- *Complete graphs.* If F is the complete graph K_p, then an instance (G_1, G_2) of the F-free graph sandwich problem has a solution if and only if G_1 is K_p-free. This can be tested in polynomial time as p is a constant.
- *Complete graph minus an edge* [6]. For every fixed p, the $(K_p \setminus e)$-free sandwich problem is in P.
- *Holes* [6]. For every fixed $k \geq 4$, the k-hole-free sandwich problem is NP-complete.
- P_4*-free graphs* (*cographs*) [15]. The P_4-free graph sandwich problem is in P.

Let F be any graph on four vertices. Up to isomorphism there are eleven such graphs. If F is either K_4 (or its complement) or $K_4 \setminus e$ (or its complement) or P_4 (which is self-complementary), then the F-free graph sandwich problem is in P by the preceding results. If F is C_4 (or its complement $2K_2$), then the problem is NP-complete as mentioned earlier. The remaining two cases are when F is the *claw* (or its complement) and the *paw* (or its complement), where the claw is the graph with vertex-set $\{a, b, c, d\}$ and edge-set $\{ab, ac, ad\}$ and the paw is the graph with vertex-set $\{a, b, c, d\}$ and edge-set $\{ab, ac, ad, bc\}$. See Fig. 1. These two cases are solved now as follows.