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Interval incidence graph coloring☆

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ABSTRACT

In this paper we introduce a concept of *interval incidence coloring* of graphs and survey its general properties including lower and upper bounds on the number of colors. Our main focus is to determine the exact value of the *interval incidence coloring number* χ_{ii} for selected classes of graphs, i.e. paths, cycles, stars, wheels, fans, necklaces, complete graphs and complete k-partite graphs. We also study the complexity of the interval incidence coloring problem for subcubic graphs for which we show that the problem of determining whether $\chi_{ii} \leq 4$ can be solved in polynomial time whereas $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete.

1. Introduction

1.1. Problem definition

In the following we consider solely simple, nonempty connected graphs with the use of the standard notation of graph theory. For a given simple graph $G = (V, E)$, we define an *incidence* as a pair (v, e), where vertex $v \in V$ is one of the end-points of edge $e \in E$. The set of all incidences of G will be denoted[1] by I, where $I := \{(v, e): v \in V \wedge e \in E \wedge v \in e\}$ and $v \in e$ means that v is one of the ends of e.[2] We say that two incidences (v, e) and (w, f) are *adjacent* if and only if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}, f = \{w, u\}$ and $v \neq u$.

By an *incidence coloring* of G we mean a function $c: I \to \mathbb{N}$ such that $c(v, e) \neq c(w, f)$ for any adjacent incidences (v, e) and (w, f). The *incidence coloring number* of G, denoted by χ_i, is the smallest number of colors in an incidence coloring of G. The incidence coloring has been well-studied [7–9] and arises from the directed star arboricity problem [1,2,15], in which one wants to partition a set of arcs into the smallest number of forests of directed stars.

A finite nonempty set $A \subseteq \mathbb{N}$ is an *interval* if and only if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring c of graph G and $v \in V$ let $A_c(v) := \{c(v, e): v \in e \wedge e \in E\}$. By an *interval incidence coloring* of graph G we mean an incidence coloring c of G such that for each vertex $v \in V$ the set $A_c(v)$ is an interval. By an interval incidence k-coloring we mean an interval incidence coloring using colors from the set $\{1, \ldots, k\}$. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring [4,11,14], which can be applied e.g. to the open-shop scheduling problem [12,13]. In [16] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [17] the authors studied applications in one-multicast transmission per vertex model in multifiber WDM networks.

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1 To simplify notation, we write I instead of $I(G)$ whenever G is clear from the context. The same rule applies to other parameters of G appearing in the paper.

2 In our definition of a graph, an edge is a set built of its ends.

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The *interval incidence coloring number* of G, denoted by χ_{ii}, is the smallest number of colors in an interval incidence coloring of G. In this paper we study the value of χ_{ii} for some classes of graphs, its bounds as well as analyze the computational complexity of the problem of determining this number.

1.2. Multicasting communication in a multifiber WDM all-optical star network

The motivation for the present paper comes from the multicasting communication in a multifiber WDM all-optical star network, which was studied in [3,5,6]. We assume that the set of n vertices V is connected to the central vertex (star network) by at most p optical parallel fibers (multifiber). The central all-optical transmitter transforms each arriving signal to the same wavelength. Each vertex $v \in V$ has to send at most q multicasts to some other vertices $S_1(v), \ldots, S_q(v)$ ($S_i(v) \subset V$). The transmission through the central vertex uses WDM (*wavelength-division multiplexing*), i.e. different signals may be simultaneously sent through the same fiber but on different wavelengths.

The first step of the multicast transmission from vertex v to $S_i(v)$ is to send a message through a fiber to the central vertex on a set of wavelengths. In the next step, the central vertex redirects the message to each vertex of $S_i(v)$ using one of these wavelengths. The goal is to minimize the total number of wavelengths used in the simultaneous transmission of all multicasts in the network. This problem can be modeled by arc coloring of labeled (multi)digraph with certain special requirements on the set of colors [3,6].

Following [3], we define a formal model and introduce a general (p, q)-WAM problem in optical star networks. Every vertex from the set V of n vertices is connected to the central vertex with p optical fibers. A simultaneous transmission of all multicasts in this network can be modeled by a (multi)digraph D with vertex set V and with labeled arc sets going out from v which correspond to multicasts to vertex sets $S_1(v), \ldots, S_{q_v}(v)$ ($q_v \leq q$), i.e. all outgoing arcs from v in set $A_i(v) = \{vw \colon w \in S_i(v)\}$ are labeled with i, for $i = 1, \ldots, q_v$.

We define a *p-fiber k-coloring* as a function assigning to each arc of digraph D a color from the set $\{1, \ldots, k\}$ such that for each vertex v and every color a, we have $inarc(v, a) + outlab(v, a) \leq p$, where $inarc(v, a)$ denotes the number of arcs entering v and colored with a, and $outlab(v, a) = |\{i \colon \exists e \in A_i(v) \text{ and where } e \text{ is colored with } a\}|$, i.e. the number of different labels of arcs outgoing v and colored with a.

Since the central vertex redirects every arriving signal to the same wavelength and different signals may be sent at the same time through the same fiber but with different wavelengths, the problem of simultaneous transmission of all multicasts in p-fiber network with the minimum number of wavelengths is equivalent to p-fiber coloring of arcs of digraph D with the minimum number of colors. Hence, we define the decision version of the problem of wavelength assignment of q-multicasts in p-fiber (optical) star networks as follows.

> The (p, q)-WAM problem: given a digraph D with at most q labels on arcs and an integer k; is there a p-fiber k-coloring of digraph D?

In the paper we consider the $(1, 1)$-WAM problem, where a fiber is unique and each vertex sends only one multicast. In this case, p-fiber coloring of multidigraph reduces to the coloring of arcs of digraph satisfying the following condition: any arc entering a vertex and any other arc at the same vertex (entering or outgoing) have different colors. This boils down to the problem of partitioning of a set of arcs into the smallest number of forests of directed stars, i.e. the previously-mentioned problem of directed star arboricity.

Let us focus our attention on the case of symmetrical communication, where every transmission from v to w implies the transmission from w and v. In this case the digraph modeling the communication between vertices is a simple graph and this problem can be reduced to the incidence coloring of graphs [7–9]. Moreover, we assume that the set of colors of incoming arcs at a vertex forms an interval. This corresponds to having consecutive wavelengths on the link between the central vertex and the destination vertex and it seems to be important for traffic grooming in WDM networks, where wavelengths could be groomed into wavebands [19].

1.3. Our results

In [18] the authors studied the problem of interval incidence coloring for subcubic bipartite graphs and trees, showing polynomial time algorithms for these classes. Moreover, they have shown that for bipartite graphs with $\Delta = 4$ the interval incidence 5-coloring is easy and 6-coloring is hard ($\mathcal{NP}$-complete).

In this paper we study the problem of interval incidence coloring for different classes of graphs, i.e. paths, cycles, stars, wheels, fans, necklaces, complete graphs and complete k-partite graphs. We focus our attention on bounding or determining the exact value of χ_{ii}. We also study the complexity of the interval incidence coloring problem for subcubic graphs for which we show that the problem of determining whether $\chi_{ii} \leq 4$ is easy, and $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete.

2. Bounds on χ_{ii}

In this section we construct certain lower and upper bounds on the interval incidence coloring number. Note that $\chi_i \leq \chi_{ii}$, hence any lower bound for χ_i is a lower bound for χ_{ii}.