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Badly-covered graphs

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We show that the maximum number of different sizes of maximal complete k-partite induced subgraphs of a graph G of sufficiently large order n is at least $(n-k+1)-\log_2(n-k+1)-4$ and at most $n-\left\lfloor \log_2\left(\frac{n}{k!}\right)\right\rfloor$. We analyze some features of the structure of graphs with the maximum possible number of different sizes of maximal independent sets, and give best-possible estimates for $K_{1,k}$-free graphs and regular graphs.

1. Introduction

In [3], Plummer defined a graph to be *well-covered* if all its maximal independent sets have the same size. At the 5th Latin American Workshop on Cliques in Graphs, Szwarcfiter [5] proposed studying graphs that are as far from being well-covered as possible; more precisely, he asked about the maximum possible number of sizes of maximal independent sets in a graph of order n. Apparently, Moon and Moser [2] were the first to contribute to this problem. Before we summarize some known results and explain our contribution, we introduce some terminology.

For a positive integer k, let $[k]$ denote the set $\{1, 2, \dots, k\}$. For a finite, simple, and undirected graph G and a positive integer k, let MMIS(G, k) denote the set of all maximal complete k-partite induced subgraphs of G. That is, for every graph H in MMIS(G, k), the vertex set $V(H)$ of H can be partitioned into k possibly empty sets $V_1, \dots, V_k$ such that

- V_i is an independent set of G for $i \in [k]$,
- G contains all edges between V_i and V_j for distinct $i, j \in [k]$, and
- H is not a proper subgraph of another complete k-partite induced subgraph of G.

Note that the elements of MMIS$(G, 1)$ correspond to the maximal independent sets of G, which in turn correspond to the cliques of the complement $\overline{G}$ of G. Let mmiss(G, k) denote the number of different orders of the graphs in MMIS(G, k); that is, the problem proposed by Szwarcfiter and already studied by Moon and Moser concerns the maximum possible value of mmiss$(G, 1)$.[1]

In 1965, Moon and Moser [2] proved that

$$\text{mmiss}(G, 1) \le n - \lfloor \log_2 n \rfloor \tag{1}$$

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1 “MMIS” stands for “maximal multipartite induced subgraphs” and “mmiss” stands for “maximal multipartite induced subgraph sizes”.

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for every graph G of order n. Furthermore, for every n, they constructed graphs G of order n with

$$\text{mmiss}(G, 1) \geq n - \lfloor \log_2 n \rfloor - 2\lfloor \log_2 \log_2 n \rfloor - 4.$$

In 1966, Erdős [1] improved the lower bound by replacing the $\log_2 \log_2 n$ term with an iterated logarithm, and, in 1971, Spencer [4] constructed for every positive integer n at least 33 000, graphs G of order n with

$$\text{mmiss}(G, 1) \geq n - \log_2 n - 4. \tag{2}$$

Even though in view of (1) and (2) the maximum value of mmiss$(G, 1)$ for graphs G of order n is known up to a small constant, the extremal graphs are not known exactly. Nevertheless, the constructions in [2,1,4] share certain recurrent features.

Our first contribution is the generalization of (1) to mmiss(G, k) for general k. Our result and a simple modification of Spencer's construction imply that the maximum values of mmiss(G, k) and mmiss$(G, 1)$ for graphs of order n only differ by a constant for fixed k. Therefore, we reconsider mmiss$(G, 1)$ and show that the recurrent features of the (almost-)extremal constructions have a reason. Finally, we give best-possible estimates for mmiss$(G, 1)$ for $K_{1,k}$-free graphs and regular graphs.

2. Results

We begin with a simple bound for general graphs, which generalizes (1).

Theorem 1. *If G is a graph of order n and k is a positive integer, then*

$$\text{mmiss}(G, k) \leq n - \left\lfloor \log_2 \left(\frac{n}{k!} \right) \right\rfloor. \tag{3}$$

Proof. Let H_0 be a complete k-partite induced subgraph of G of maximum order $n(H_0)$. If $n(H_0)$ is at most the right-hand side of (3), then (3) follows immediately. Hence, we may assume that the set $R = V(G) \setminus V(H_0)$ has order at most $\log_2 \left(\frac{n}{k!}\right) - 1$. Let $(V_1, \ldots, V_k)$ be an ordered partition of $V(H_0)$ into possibly empty independent sets that are pairwise completely joined in G.

Let S be a subset of R. Note that there are $2^{|R|}$ such subsets. We claim that there are at most $k!$ distinct graphs H in MMIS(G, k) with $S = V(H) \cap R$. In fact, if $G[S]$ is not a complete k-partite graph, then there is no such graph. Hence we may assume that $G[S]$ is a complete k-partite graph. Let $\mathcal{P} = (S_1, \ldots, S_k)$ be an ordered partition of S into possibly empty independent sets that are pairwise completely joined in G. Note that there are at most $k!$ such ordered partitions. If, for $i \in [k]$,

$$V_i' = \{u \in V_i : N_G(u) \cap S_i = \emptyset \text{ and } S_j \subseteq N_G(u) \text{ for every } j \in [k] \setminus \{i\}\},$$

then $H(\mathcal{P}) = G\left[\bigcup_{i=1}^k S_i \cup V_i'\right]$ is the unique maximal complete k-partite induced subgraph of G that arises from $G[S]$ by adding, for every $i \in [k]$, vertices from V_i to S_i.

Now, let H be a graph in MMIS(G, k) with $S = V(H) \cap R$. Let $(U_1, \ldots, U_k)$ be an ordered partition of $V(H)$ into possibly empty independent sets that are pairwise completely joined in G. Since H and H_0 are complete k-partite, every set U_i intersects at most one of the sets V_j, and every set V_j intersects at most one of the sets U_i. This implies that we may assume that $U_i \cap V_j = \emptyset$ for $i, j \in [k]$ with $i \neq j$; that is, we assume that the partition $(U_1, \ldots, U_k)$ is ordered in such a way that the set U_i intersects at most the set V_i for $i \in [k]$. Let $T_i = U_i \cap R$ for $i \in [k]$. Note that $(T_1, \ldots, T_k)$ is an ordered partition of S into possibly empty independent sets that are pairwise completely joined in G. In view of the above remark about the uniqueness of $H(\mathcal{P})$, it follows that $H = H((T_1, \ldots, T_k))$; that is, each graph in MMIS(G, k) equals $H(\mathcal{P})$ for some ordered partition $\mathcal{P}$ of S.

Since $|R| \leq \log_2 \left(\frac{n}{k!}\right) - 1$, this implies that mmiss$(G, k) \leq k!2^{|R|} \leq \frac{n}{2} \leq n - \left\lfloor \log_2 \left(\frac{n}{k!}\right) \right\rfloor$, which completes the proof. □

Adding to the graphs constructed by Spencer [4] an independent set of $k - 1$ vertices joined to all remaining vertices yields graphs G of order $n \geq 33\,000 + (k - 1)$ with mmiss$(G, k) \geq (n - k + 1) - \log_2(n - k + 1) - 4$; that is, for constant k and sufficiently large order, the bound in Theorem 1 is best possible up to a constant additive term.

We now reconsider the extremal graphs for mmiss$(G, 1)$. For $k = 1$, the graph H_0 considered in the proof of Theorem 1 is a maximum independent set. In order to receive essentially $n - \lfloor \log_2 n \rfloor$ maximal independent sets of different sizes, the independence number $\alpha(G)$ of G should be essentially equal to $n - \lfloor \log_2 n \rfloor$; that is, the complement R should have order essentially equal to $\lfloor \log_2 n \rfloor$. Furthermore, for essentially every subset S of R, the graph G should contain a maximal independent set $I(S)$ with $I(S) \cap R = S$, and essentially all these sets should be of different size. As we show in Theorem 3 below, this extreme situation naturally leads to the disjoint union of exponentially growing stars, which is a common constructive feature in all (almost-)extremal constructions [2,1,4].

Before we come to this result, we need a small number-theoretic lemma. We have the feeling that this lemma should be well known, but since we did not find a reference, we include the simple proof.

Lemma 2. *If, for $k \in \mathbb{N}$, D is a set of k positive integers such that every integer between 0 and $2^k - 1$ has a representation as a sum of elements of D, then $D = \{2^0, 2^1, \ldots, 2^{k-1}\}$.*

Proof. We prove the statement by induction on k. For $k = 1$, the result is trivial. Now, let $k \geq 2$. Let $D = \{d_1, \ldots, d_k\}$ be such that $d_1 \leq d_2 \leq \cdots \leq d_k$. Since D has 2^k subsets and there are exactly 2^k integers between 0 and $2^k - 1$, every such integer has a unique representation as a sum of elements of D, and all sums of the elements of distinct subsets of D are distinct.

If $d_k > 2^{k-1}$, then every integer between 0 and 2^{k-1} has a representation as a sum of elements of $D \setminus \{d_k\}$. Since there are $2^{k-1} + 1$ such integers but $D \setminus \{d_k\}$ only has 2^{k-1} subsets, this is impossible.