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Determining the optimal strategies for discrete control problems on stochastic networks with discounted costs

Dmitrii Lozovanu [a], Stefan Pickl [b,*]

[a] Institute of Mathematics and Computer Science, Academy of Sciences of Moldova, Academy str., 5, Chisinau, MD-2028, Republic of Moldova

[b] Institute for Theoretical Computer Science, Mathematics and Operations Research, Universität der Bundeswehr München, 85577 Neubiberg-München, Germany

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ABSTRACT

The main results of the paper are concerned with determining the optimal stationary strategies for stochastic discrete control problems on networks with discounted costs. We ground polynomial time algorithms for determining the optimal strategies of this problem using a linear programming approach. Additionally, we show that the proposed approach can be extended for Markov decision processes with a total discounted cost optimization criterion.

1. Introduction and problem formulation

The stochastic discrete control problems and Markov decision processes with discounted costs have been studied in [5,6,10,12]. Investigations concerned with the elaboration of numerical algorithms for solving these problems arise in many practical decision problems from diverse areas such as ecology, economics, engineering, communications systems etc. (see [1,2,6,10,11]). Efficient iterative procedures for determining the optimal stationary strategies for infinite horizon control problems and a Markov decision problem with discounted costs are described in [5–13]. In this paper we consider the infinite horizon stochastic control problem on networks with discounted costs and develop a linear programming approach for determining its optimal solutions. Afterwards we extend the linear programming approach for a Markov decision problem. The formulation of the infinite horizon control problem on networks is the following.

Let a time-discrete system $\mathbb{L}$ with a finite set of states X be given. At every discrete moment of time $t = 0, 1, 2, \ldots$ the state of the dynamical system is given by $x(t) \in X$. The dynamics of the system is described by a directed graph of states' transitions $G = (X, E)$, where the vertex set X corresponds to the set of states of $\mathbb{L}$, and an arbitrary directed edge $e = (x, y) \in E$ expresses the possibility of the dynamical system to pass from the state $x = x(t)$ to the state $y = x(t)$ at every discrete moment of time $t = 0, 1, 2, \ldots$. On the edge set E a cost function $c : E \to \mathbb{R}$ is defined that gives a cost c_e to each directed edge $e = (x, y) \in E$ when the system makes a transition from the state $x = x(t)$ to the state $y = x(t + 1)$ for every $t = 0, 1, 2, \ldots$. We define a stationary control for the system $\mathbb{L}$ in G as a map

$$s : x \mapsto y \in X(x) \quad \text{for } x \in X,$$

where $X(x) = \{y \in X \mid (x, y) \in E\}$.

* Corresponding author. Tel.: +49 8960042400.
E-mail addresses: lozovanu@math.md (D. Lozovanu), stefan.pickl@unibw.de (S. Pickl).

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Let s be an arbitrary stationary control. Then the set of edges of the form $(x, s(x))$ in G generates a subgraph $G_s = (X, E_s)$ where each vertex $x \in X$ contains one leaving directed edge. So, if the starting state $x_0 = x(0)$ is fixed then the system makes transitions from one state to another through the corresponding directed edges $e_0^s, e_1^s, e_2^s, \dots, e_t^s, \dots$, where $e_t^s = (x(t), x(t+1)),\ t = 0, 1, 2, \dots$. This sequence of directed edges generates a trajectory $x_0 = x(0), x(1), x(2), \dots$ which leads to a unique directed cycle. For an arbitrary stationary strategy s and a fixed starting state x_0 the discounted expected total cost $\sigma_{x_0}^{\gamma}(s)$ is defined as follows

$$\sigma_{x_0}^{\gamma}(s) = \sum_{t=0}^{\infty} \gamma^t c_{e_t^s},$$

where $\gamma,\ 0 < \gamma < 1$, is a given discount factor.

Based on the results from [2,10,12] it is easy to show that for an arbitrary stationary strategy s there exists $\sigma_{x_0}^{\gamma}(s)$. If we denote by $\sigma^{\gamma}(s)$ the column vector with components $\sigma_x^{\gamma}(s)$ for $x \in X$ then $\sigma_{x_0}^{\gamma}(s)$ can be found by solving the system of linear equations

$$(I - \gamma P^s)\sigma^{\gamma}(s) = c^s, \tag{1}$$

where c^s is the vector with corresponding components $c_{x,s(x)}$ for $x \in X$, I is the identity matrix and P^s the matrix with elements $p_{x,y}^s$ for $x, y \in X$ defined as follows:

$$p_{x,y}^s = \begin{cases} 1, & \text{if } y = s(x); \\ 0, & \text{if } y \neq s(x). \end{cases}$$

It is well known that for $0 < \gamma < 1$ the rank of the matrix $I - \gamma P^s$ is equal to $|X|$ and system (1) has solutions for arbitrary c^s (see [10,12]). Thus, we can determine $\sigma_{x_0}^{\gamma}(s^*)$ for an arbitrary starting state x_0.

In the considered deterministic discounted control problem on G we are seeking for a stationary control s^* such that

$$\sigma_{x_0}^{\gamma}(s^*) = \min_s \sigma_{x_0}^{\gamma}(s).$$

We formulate and study this problem in a more general case considering its stochastic version. We assume that the dynamical system may admit states in which the vector of control parameters is changed in a random way. So, the set of states X is divided into two subsets $X = X_C \cup X_N,\ X_C \cap X_N = \emptyset$, where X_C represents the set of states in which the decision maker is able to control the dynamical system and X_N represents the set of states in which the dynamical system makes transition to the next state in a random way. So, for every $x \in X$ on the set of feasible transitions $E(x)$ the distribution function $p : E(x) \mapsto \mathbb{R}$ is defined such that $\sum_{e \in E(x)} p_e = 1,\ p_e \geq 0,\ \forall e \in E(x)$ and the transitions from the states $x \in X_N$ to the next states are made according to these distribution functions. Here, in a similar way as for the deterministic problem, we assume that to each directed edge $e = (x, y) \in E$ a cost c_e of system's transition from the state $x = x(t)$ to the state $y = x(t+1)$ for every $t = 0, 1, 2, \dots$ is associated. In addition we assume that the discount factor $\gamma,\ 0 < \gamma < 1$, and the starting state x_0 are given. We define a stationary control on G as a map

$$s : x \mapsto y \in X(x) \quad \text{for } x \in X_C.$$

Let s be an arbitrary stationary strategy. We define the graph $G_s = (X,\ E_s \cup E_N)$, where $E_s = \{e = (x, y) \in E \mid x \in X_C,\ y = s(x)\},\ E_N = \{e = (x, y) \mid x \in X_N,\ y \in X\}$. This graph corresponds to a Markov process with the probability matrix $P^s = (p_{x,y}^s)$, where

$$p_{x,y}^s = \begin{cases} p_{x,y}, & \text{if } x \in X_N \text{ and } y = X; \\ 1, & \text{if } x \in X_C \text{ and } y = s(x); \\ 0, & \text{if } x \in X_C \text{ and } y \neq s(x). \end{cases}$$

For this Markov process with associated costs $c_e,\ e \in E$ we can define the expected total discounted cost $\sigma_{x_0}^{\gamma}(s)$ as we have introduced it in the first paragraph. We consider the problem of determining the strategy s^* for which

$$\sigma_{x_0}^{\gamma}(s^*) = \min_s \sigma_{x_0}^{\gamma}(s).$$

Without loss of generality we may consider that G poses the property that an arbitrary vertex in G is reachable from x_0; otherwise we can delete all vertices that could not be reached from x_0.

2. A linear programming approach

Let s be an arbitrary stationary strategy. We identify this strategy with the set of boolean variables $s_{x,y}$ for $x \in X_C$ and $y \in X(x)$, where

$$s_{x,y} = \begin{cases} 1, & \text{if } y = s(x); \\ 0, & \text{if } y \neq s(x). \end{cases} \tag{2}$$

In the following we will simplify the notations and instead of σ_x^{γ} we shall use σ_x.