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Combinatorial optimization with one quadratic term: Spanning trees and forests ☆

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ABSTRACT

The standard linearization of a binary quadratic program yields an equivalent reformulation as an integer linear program, but the resulting LP-bounds are very weak in general. We concentrate on applications where the underlying linear problem is tractable and exploit the fact that, in this case, the optimization problem is still tractable in the presence of a single quadratic term in the objective function. We propose to strengthen the standard linearization by the use of cutting planes that are derived from jointly considering each single quadratic term with the underlying combinatorial structure. We apply this idea to the quadratic minimum spanning tree and spanning forest problems and present complete polyhedral descriptions of the corresponding problems with one quadratic term, as well as efficient separation algorithms for the resulting polytopes. Computationally, we observe that the new inequalities significantly improve dual bounds with respect to the standard linearization, particularly for sparse graphs.

1. Introduction

Optimization problems with a quadratic objective function and linear constraints over binary variables are usually hard to solve. This remains true in general in the special case where the underlying linear problem is polynomially solvable; even in the unconstrained case the binary optimization problem is NP-hard due to its equivalence to the Maximum-Cut problem [5].

A very common approach to binary quadratic optimization is to linearize the quadratic terms and to develop an appropriate polyhedral description of the corresponding set of feasible solutions. For reasons of complexity, one cannot expect to find a tight and polynomial sized polyhedral description. A straightforward idea is to linearize each product in the objective function independently and simply combine the result with the given linear side constraints [7]. This approach yields a correct integer programming model of the problem, but the resulting LP-relaxations lead to very weak bounds in general, so that branch-and-cut algorithms based on this simple linearization idea perform very poorly in practice. For this reason, one usually searches for stronger inequalities to tighten the description; see, e.g., [3].

In this paper, we consider another approach that, to the best of our knowledge, has not been investigated yet: we examine the problem version with only one product term in the objective function, but with all linear side constraints taken into account [2]. Any valid cutting plane for this problem will remain valid for the original problem and potentially improves over the straightforward model, since the chosen product is considered together with all side constraints. The advantage of our approach lies in the fact that there exists a polynomial time separation algorithm for the one-product problem whenever the underlying linear problem is tractable. This is guaranteed by a general result by Grötschel et al. [10], since the corresponding

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optimization problem is polynomially solvable: for this, we can solve the underlying linear problem four times, with different fixings of the two variables appearing in the chosen product. The resulting cutting planes may not be facet defining for the original problem, but they lead to a tighter polyhedral description in general.

From a practical point of view, the indirect separation approach by Grötschel et al. is too general and it does not pay off to apply it inside a branch-and-cut approach; the computational effort for generating a single cutting plane is too high. Therefore, we have a closer look at specific quadratic optimization problems in order to find valid inequalities and fast separation routines for the quadratic case, which might be inspired by the corresponding techniques in the linear case.

In this paper, we investigate our approach for the quadratic minimum spanning tree (QMST) problem and the closely related quadratic minimum spanning forest (QMSF) problem. The *linear* spanning forest problem deals with finding a cycle free spanning subgraph of minimal cost in a given underlying graph, where costs are defined edge-wise. If additionally the subgraph is required to be connected, we obtain the linear version of the spanning tree problem. The latter problem is well studied and solvable in polynomial time, e.g., by the algorithms of Prim [17] or Kruskal [12]. If additional costs arise for pairs of edges, we obtain the QMST problem, which was shown to be NP-hard in general by Assad and Xu [1]. In their work the authors also give examples for the QMST problem arising in applications related to transportation problems or communication and energy networks. Here, additional costs occur whenever two adjacent edges of different types are chosen to belong to the tree. Examples are trees with changeover and reload costs, recently studied by Galbiati et al. [8,9]. Exact and heuristic algorithms for QMST have been presented by Cordone and Passeri [4], Öncan and Punnen [15], and very recently by Pereira et al. [16], who apply the reformulation–linearization technique (RLT) combined with Lagrangian relaxation. More heuristic approaches are discussed in [22,21,14]. Special cases of QMST have been shown to be tractable, e.g., the case of a multiplicative objective function with positive factors [18].

In contrast to the QMST, the quadratic minimum *forest* problem has received little attention in the literature so far. Lee and Leung [13] define the polytope corresponding to the linearized QMSF as the *Boolean Quadric Forest Polytope* and classify several facet classes. For some of these classes, they develop polynomial time separation algorithms. Note that many variants of the linear spanning forest problem have been considered in the literature, e.g., the number of connected components or the degree of vertices may be bounded. In this paper, we do not consider any such restriction but optimize over the set of all spanning forests in the given graph.

Following our general approach outlined above, we devise complete polyhedral descriptions of the spanning forest problem and the spanning tree problem with one quadratic term. It turns out that, beyond the well-known subtour elimination constraints for the linear case and the standard linearization constraints, only one additional exponential-size class of facet-defining inequalities is needed for the complete description. The shape of these constraints depends on whether the two edges involved in the quadratic term share a vertex or not; in both cases they are closely related to the subtour elimination constraints. In particular, we present exact and efficient separation algorithms based on the separation algorithm for subtour elimination constraints.

Turning back to the general QMSF and QMST problems, this separation algorithm can be applied to each quadratic term independently. We evaluate the strength of the resulting relaxation by computational experiments, integrating the new separation algorithm into a branch-and-bound scheme. Our experiments show that the new cutting planes significantly improve over the standard linearization in terms of dual bounds, particularly for sparse graphs.

Based on our approach, Fischer and Fischer [6] recently published an alternative proof for one of our results that devises a complete polyhedral description of the spanning tree problem with one quadratic term. For showing optimality of a given tree with respect to this linear description, both proofs construct a corresponding dual solution. While we construct the dual solution directly, distinguishing the four possible assignments of the two product variables, Fischer and Fischer start from a dual optimal solution obtained by ignoring the quadratic term and adapt this solution appropriately. However, they do not discuss separation algorithms and, in particular, they do not evaluate the approach experimentally.

This paper is organized as follows. In Section 2, we introduce the basic notation needed throughout the paper. Sections 3 and 4 present our polyhedral results for QMSF and QMST with one quadratic term. The corresponding separation problem is addressed in Section 5, whereas Section 6 contains the results of an experimental evaluation. Section 7 concludes.

2. Preliminaries

Throughout this paper, we assume that $G = (V, E)$ is a complete undirected graph. A *spanning forest* F is a cycle free subgraph of G with $V(F) = V$. In a weighted graph, the cost of a spanning forest is the sum of edge weights c_e over all edges $e \in E(F)$. If additional costs q_{ef} arise for each pair of different edges $e, f \in E$ contained in the forest, we have a *quadratic minimum spanning forest problem* (QMSF). In a very natural way, QMSF can be formulated as an integer program with linear constraints and a quadratic objective function:

$$(\text{QIP}_{\text{QMSF}}) \qquad \min \sum_{e\in E} c_e x_e + \sum_{\substack{e,f\in E\\ e\neq f}} q_{ef} x_e x_f$$

$$\text{s.t.} \sum_{e\in E(G[S])} x_e \le |S| - 1 \qquad \forall \emptyset \neq S \subseteq V \tag{1}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E.$$