Discrete Applied Mathematics 209 (2016) 133–136

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Making a C_6-free graph C_4-free and bipartite

Ervin Győri [a,b,*], Scott Kensell [b], Casey Tompkins [b]

[a] MTA Rényi Institute, Hungarian Academy of Sciences, Budapest, Hungary

[b] Department of Mathematics, Central European University, Budapest, Hungary

ARTICLE INFO

Article history:
Received 13 November 2014
Received in revised form 9 June 2015
Accepted 15 June 2015
Available online 13 July 2015

Keywords:
Graph theory
Extremal graphs
Bipartite subgraphs
6-cycles
4-cycles

ABSTRACT

We show that every C_6-free graph G has a C_4-free, bipartite subgraph with at least $3e(G)/8$ edges. Our proof is probabilistic and uses a theorem of Füredi et al. (2006) on C_6-free graphs.

1. Introduction

For a graph G, let $e(G)$ denote the number of edges in G. We say G is H-free if it does not contain H as a subgraph. For a family of graphs $\mathcal{F}$, let $\mathrm{ex}(n, \mathcal{F})$ denote the maximum number of edges an n-vertex graph G can have such that G is F-free for all $F \in \mathcal{F}$.

Győri [2] proved that every bipartite, C_6-free graph contains a C_4-free subgraph with at least half as many edges. Extending this result, Kühn and Osthus [3] showed that every bipartite, C_{2k}-free graph has a C_4-free subgraph with at least $1/(k-1)$ of the original edges. In an extensive study of the Turán number $\mathrm{ex}(n, C_6)$, Füredi, Naor and Verstraëte [1] gave another generalization of Győri's result by showing (Theorem 3.1) that a C_6-free graph has a triangle-free, C_4-free subgraph with at least half as many edges.

Using any of these results combined with the well-known fact that every graph has a bipartite subgraph with at least half as many edges, it is easy to show that any C_6-free graph has a bipartite, C_4-free subgraph with at least 1/4 the original edges. Improving the constant 1/4 is the main focus of this paper.

In general, if we would like to make a C_6-free graph C_4-free and bipartite, we cannot hope to keep more than 2/5 of its edges (consider many disjoint K_5's). We show that if c is the maximum constant such that every C_6-free graph G has a C_4-free subgraph on $c \cdot e(G)$ edges then $3/8 \leq c \leq 2/5$.

Theorem 1. *Let G be a C_6-free graph, then G contains a subgraph with at least $3e(G)/8$ edges which is both C_4-free and bipartite.*

The result can also be phrased in the language of Turán theory: If $\mathcal{C}$ denotes the set of all odd cycles, then $\mathrm{ex}(n, C_6) \leq 8\,\mathrm{ex}(n, C_4, C_6, \mathcal{C})/3$.

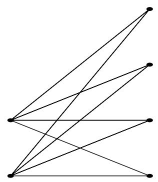
* Corresponding author at: MTA Rényi Institute, Hungarian Academy of Sciences, Budapest, Hungary. Tel.: +36 1 4838302; fax: +36 1 4838333.
E-mail addresses: gyori.ervin@renyi.mta.hu (E. Győri), skensell@gmail.com (S. Kensell), ctompkins496@gmail.com (C. Tompkins).

http://dx.doi.org/10.1016/j.dam.2015.06.008
0166-218X/

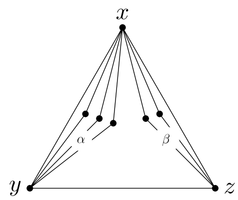
Our proof is a probabilistic deletion procedure consisting of several steps. First we two-color the vertices, and then, focusing on specific edge-disjoint subgraphs, we delete certain edges given the outcome of the coloring. These edge-disjoint subgraphs are the maximal subgraphs obtained by pasting together edge-intersecting C_4's and were characterized by Füredi, Naor and Verstraëte. We use the following slightly weaker formulation of their theorem.

Theorem 2. *For a C_6-free graph G, let G^* denote the graph whose vertex set is the collection of C_4's in G and whose edge set represents edge-intersection. Then each connected component of G^* corresponds to an induced subgraph H of G of one of the following types:*

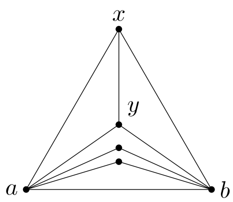
(0) *the complete bipartite graph $K_{2,m}$ for some $m > 0$,*
(1) *a triangle xyz with α additional vertices adjacent to x and y but not z, and β more vertices adjacent to x and z but not y,*
(2) *a K_4 with $\gamma \geq 0$ additional paths of length 2 between a fixed pair of its vertices,*
(3) *a K_5, K_5 minus an edge, or a K_5 minus two non-adjacent edges.*



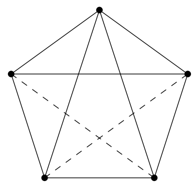
Type 0



Type 1



Type 2



Type 3

2. Proof of Theorem 1

Independently at random, color all vertices in G red or blue with probability $1/2$ each. Deleting all monochromatic edges would yield a bipartite graph, but some C_4's may remain. Thus, given the random coloring we will deterministically delete additional edges in such a way that, upon deletion of monochromatic edges, at least $3e(G)/8$ edges remain in expectation, but all C_4's are deleted. Notice that after coloring, the C_4's which require further edge deletion are exactly the properly colored C_4's (those with no monochromatic edges).

For each component H of type 0, 1, 2, or 3 from Theorem 2 we will show that our vertex-coloring and subsequent edge-deletion procedure preserves at least $3e(H)/8$ edges in expectation. Since these components are edge-disjoint and cover all C_4's, we are then done by linearity of expectation.

Case(*H* **is the null graph**): G is C_4-free so the result is immediate.

Case(*H* **is of type 0**): First, suppose H is a component of type 0. That is, H is a complete bipartite graph $K_{2,t}$. Let x and y be the vertices in the first class, and $v_1, v_2, \ldots, v_t$ be the vertices in the second class. If x and y are opposite colors, then there are no properly colored C_4's, and the expected number of remaining edges is exactly $e(H)/2$.

Now, suppose that x and y are the same color, say red. If none of the v_i's are colored blue then we lose all edges in H. If exactly s, $s \geq 1$, of the v_i's are colored blue, then we must delete all but one of the edges emanating from x to the v_i's for otherwise we would have a properly colored C_4. Thus, exactly $s+1$ edges will remain in H. The probability that s of the v_i are blue is $\binom{t}{s}/2^t$. Let N_0 be the random variable equal to the number of edges which remain in H, then

$$\begin{aligned}\mathbb{E}(N_0 \mid x \text{ and } y \text{ same color}) &= \frac{1}{2^t}0 + \sum_{s=1}^{t} \frac{\binom{t}{s}}{2^t}(s+1) \\ &= \frac{1}{2^t}\sum_{s=1}^{t}\binom{t}{s}s + \frac{1}{2^t}\sum_{s=1}^{t}\binom{t}{s} \\ &= \frac{1}{2^t}t2^{t-1} + \frac{1}{2^t}(2^t-1) \\ &\geq \frac{t}{2} + \frac{1}{2}.\end{aligned}$$