Discrete Applied Mathematics 209 (2016) 247–250

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Sufficient conditions for 2-rainbow connected graphs

Arnfried Kemnitz [a], Ingo Schiermeyer [b,*]

[a] Computational Mathematics, Technische Universität Braunschweig, 38023 Braunschweig, Germany

[b] Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, 09596 Freiberg, Germany

A R T I C L E I N F O

Article history:
Received 27 September 2014
Received in revised form 12 October 2015
Accepted 15 October 2015
Available online 11 November 2015

Keywords:
Rainbow colouring
Rainbow connected

A B S T R A C T

An edge-coloured connected graph G is rainbow connected if each two vertices are connected by a path whose edges have distinct colours. If such a colouring uses k colours then G is called k-rainbow connected. The rainbow connection number of G, denoted by $\mathrm{rc}(G)$, is the minimum k such that G is k-rainbow connected. Even the problem to decide whether $\mathrm{rc}(G) = 2$ is NP-complete. It has been shown that if G is a connected graph of order n and size m with $\binom{n-2}{2} + 2 \leq m \leq \binom{n-1}{2}$, then $2 \leq \mathrm{rc}(G) \leq 3$.

In this paper we will present sufficient conditions for graphs G of this size to fulfil $\mathrm{rc}(G) = 2$ depending on vertex degrees.

1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured connected graph G is called *rainbow connected* if each two vertices are connected by a path whose edges have different colours. Note that the edge colouring need not be proper. If such a colouring uses k colours then G is called *k-rainbow connected*. The rainbow connection number of G, denoted by $\mathrm{rc}(G)$, is the minimum k such that G is k-rainbow connected.

This concept of rainbow connection in graphs was introduced by Chartrand et al. in [4]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may colour the edges of a given spanning tree of G with different colours and colour the remaining edges with one of the already used colours. Chartrand et al. [4] determined the precise rainbow connection number of several graph classes including complete multipartite graphs. The rainbow connection number has been studied for other graph classes in [2] and for graphs with fixed minimum degree in [2,8,11].

The computational complexity of rainbow connectivity has been studied in [3,9]. It is proved that the computation of $\mathrm{rc}(G)$ is NP-hard [3,9]. In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$. It is also NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow connected [3]. More generally, it has been shown in [9] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection number of graphs the following results are known (and obvious).

Proposition 1. *Let G be a connected graph of order n. Then*

1. $1 \leq \mathrm{rc}(G) \leq n - 1$,
2. $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,
3. $\mathrm{rc}(G) = 1$ *if and only if* G *is complete,*
4. $\mathrm{rc}(G) = n - 1$ *if and only if* G *is a tree.*

* Corresponding author.
E-mail addresses: a.kemnitz@tu-bs.de (A. Kemnitz), Ingo.Schiermeyer@tu-freiberg.de (I. Schiermeyer).

http://dx.doi.org/10.1016/j.dam.2015.10.025
0166-218X/

Let $f(n, k)$ be the minimum integer such that every connected graph of order n and of size at least $f(n, k)$ is k-rainbow connected. In [7] the following problem was introduced:

Problem 1. Determine $f(n, k)$ for all n and k with $1 \le k \le n-1$.

The following result is stated and proved in [7], but we reproduce the proof here for the readers' convenience.

Proposition 2. *For n and k with $1 \le k \le n-1$ it holds that $f(n, k) \ge \binom{n-k+1}{2} + k - 1$.*

Proof. We construct a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n-k-1$ with u_1 and u_2. Now take a path P_k with vertices labelled $w_1, w_2, \ldots, w_k$ and identify the vertices u_2 and w_1. The resulting graph G_k has order n and size $|E(G_k)| = \binom{n-k+1}{2} + k - 2$. For its diameter we obtain $d(u_1, w_k) = \mathrm{diam}(G_k) = k + 1$ which implies $\mathrm{rc}(G_k) \ge k + 1$. Hence $f(n, k) \ge \binom{n-k+1}{2} + k - 1$. □

This leads to our second problem.

Problem 2. Determine all values of n and k such that $f(n, k) = \binom{n-k+1}{2} + k - 1$.

It has been shown that $f(n, k) = \binom{n-k+1}{2} + k - 1$

- for $k = 1, 2, n-2$, and $n-1$ in [7],
- for $k = 3$ and 4 in [10],
- and for $n-6 \le k \le n-3$ in [6].

Moreover, the following theorem was proved in [7].

Theorem 1 ([7]). *Let G be a connected graph of order n and size m. If $\binom{n-1}{2} + 1 \le m \le \binom{n}{2} - 1$, then $\mathrm{rc}(G) = 2$.*

From the above mentioned results we deduce that if G is a connected graph of order n and size m with $\binom{n-2}{2} + 2 \le m \le \binom{n-1}{2}$, then $2 \le \mathrm{rc}(G) \le 3$. In this paper we will present sufficient conditions for graphs G of size in this range fulfilling $\mathrm{rc}(G) = 2$. The conditions depend on vertex degrees of the graph. Note that, as mentioned earlier, the problem to decide whether $\mathrm{rc}(G) = 2$ is NP-complete.

2. Recursive approaches

In this section we will prove sufficient conditions for graphs to be 2-rainbow connected. The conditions are based on recursions. The following lemma will be used in proving the results.

Lemma 1. *Let G be a connected graph of order n and $u \in V(G)$ be a vertex with $d(u) \ge \frac{n-1}{2}$. Let $H = G - u$, $T = V(G) \setminus N[u]$, and $t = |T|$. Suppose that $\mathrm{diam}(G) = 2$ and there is a perfect t-matching (tK_2) between all vertices of T and $N(u)$. Then it holds: If H is 2-rainbow connected, then G is 2-rainbow connected.*

Proof. Let $T = \{w_1, w_2, \ldots, w_t\}$ and $N(u) = \{v_1, v_2, \ldots, v_{d(u)}\}$. We may assume that $v_i w_i \in E(G)$ for $1 \le i \le t$. Consider now a 2-rainbow colouring c of the edges of H with colours 1 and 2. Then we define $c(uv_i) = 3 - c(v_i w_i)$ for $1 \le i \le t$. Since H is 2-rainbow connected, we have to check that there is a 2-rainbow path between u and v for all vertices $v \in V(G) \setminus \{u\}$. If $v \in N(u)$, then this is the case. If $v = w_i$ with $1 \le i \le t$, then uv_iw_i is a 2-rainbow path. Hence, G is 2-rainbow connected. □

Theorem 2. *Let G be a connected graph of order n and size m with $\mathrm{diam}(G) = 2$ and $\binom{n-2}{2} + 2 \le m \le \binom{n-1}{2}$. If $m \ge \binom{n-2}{2} + d(u) + 1$ for a vertex u with $d(u) \ge \frac{n}{2}$, then $\mathrm{rc}(G) = 2$.*

Proof. Let $H = G - u$, $T = V(G) \setminus N[u]$, and $t = |T|$. Then $|E(H)| \ge \binom{n-2}{2} + 1$ and so $1 \le \mathrm{rc}(H) \le 2$ by Theorem 1 and Proposition 1.

We now show that there is a t-matching between the vertices of T and $N(u)$. Suppose there is no such t-matching. Then the marriage theorem of Hall [5] implies that there exists a subset $S \subseteq T$ with $|S| = k$ for some k with $2 \le k \le t$ such that $|N(S) \cap N(u)| \le k - 1$ (note that $k = 1$ implies $\mathrm{diam}(G) \ge 3$, a contradiction). Since $d(u) \ge \frac{n}{2}$, we have $2 \le t \le \frac{n-2}{2}$. Hence $|E(N(u), T)| \le t(n-1-t) - k(n-1-t-(k-1)) = t(n-1-t) - k(n-t-k)$ where $E(N(u), T)$ is the set of edges between $N(u)$ and T. So the number of missing edges between $N(u)$ and T is at least $k(n-t-k)$. We set $g(k) = k(n-t-k)$. Then $g(k)$ is concave in k since the second derivative is -2 and therefore $g(k) \ge \min\{g(2), g(t)\} \ge n-2$ for all $2 \le k \le t \le \frac{n-2}{2}$, where $g(2) \ge n-2$ and $g(t) \ge g(\frac{n-2}{2}) = n-2$. Since $E(N(u), T) \subseteq E(H)$ and at least $g(k)$ edges are missing from H, we obtain $|E(H)| \le \binom{n-1}{2} - (n-2) = \binom{n-2}{2} < \binom{n-2}{2} + 1$, a contradiction. Applying Lemma 1 implies that G is 2-rainbow connected. □