Discrete Applied Mathematics 206 (2016) 211–214

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

On the geodetic iteration number of the contour of a graph

Mauro Mezzini

Department of Educational Sciences, Roma Tre University, Italy

ARTICLE INFO

Article history:
Received 17 September 2015
Received in revised form 1 December 2015
Accepted 13 February 2016
Available online 9 March 2016

Keywords:
Geodetic iteration number
Contour of a graph
Geodesic convexity

ABSTRACT

Let G be a graph and S be a subset of vertices of G. With $I[S]$ we denote the set of all vertices on some geodesic (shortest path) between two vertices of S. A contour vertex of a graph is one whose eccentricity is at least as big as all its neighbors' eccentricities. Let C be the set of contour vertices of a graph. We provide the first example of a graph where $I[I[C]]$ do not coincide with the vertex set of the graph.

1. Introduction

The study of abstract convexity leads to the generalization of the classical concept of a Euclidean convex set to finite structures such as graphs. Among the various notions of convexity in graphs, geodesic convexity, based on the concept of shortest path, is the most widely studied [7,8,6].

A *geodesic* is a shortest path between a pair of vertices of a graph G. The *interval* $I[u, v]$ is the set of all vertices lying in some geodesic between u and v. Given a set of vertices S, $I[S]$ is defined as

$$I[S] = \bigcup_{\{u,v\}\subseteq S} I[u, v].$$

Furthermore we denote by $I^k[S]$, $I[S]$ if $k = 1$ and $I[I^{k-1}[S]]$ if $k > 1$, k integer. A subset S of vertices of a graph is said to be *convex* if $S = I[S]$. If S is not convex the *convex hull* of S is the set S_k obtained at the end of the following sequence: $S_0 = S$, $S_1 = I[S], \dots, S_i = I^i[S]$. The sequence ends when $S_k = I^{k+1}[S]$ and the minimum k for which the above sequence ends, is called the *geodetic iteration number* of S, denoted as $gin(S)$ [4]. A set S of vertices is *geodetic* if its interval is the vertex set of G, that is if $I[S] = V(G)$. The *eccentricity* of a vertex v, denoted as $ecc(v)$, is the maximum distance of v from any other vertex of G. A *contour vertex* of G is a vertex v whose eccentricity is greater or equal to that of every vertex adjacent to v. The *contour* of G is the set of contour vertices of $V(G)$, denoted as $Ct(G)$.

Cáceres et al. [3] showed that the convex hull of the contour of a graph does coincide with the vertex set of the graph. Hence $gin(Ct(G))$ is the minimum k such that $I^k[Ct(G)] = V(G)$. In the same paper the authors showed that if the graph is distance-hereditary, every convex set can be "rebuilded" from its contour vertices by applying the interval operation, that is: the contour of a distance-hereditary graph is geodetic. Later on, a number of works in literature focused on determining for which classes of graphs the set of contour vertices is geodetic. It was proved that, for chordal graphs [2], for 3-Steiner distance hereditary graphs [5], for HHD-graphs [5,9], for cochordal graphs [1] and for cactus [9] the contour is geodetic. More recently it has been determined that the contour is geodetic also for the class of bridged graphs [10].

In [2,1,9], it has been shown the existence of graphs for which $I[Ct(G)] \neq V(G)$. However, the question of whether the $gin(Ct(G)) \leq 2$ remained open for more than a decade. In this paper we will provide a negative answer to the above question by showing a graph in which $I^2[Ct(G)] \neq V(G)$.

E-mail address: mezzini@di.uniroma1.it.

http://dx.doi.org/10.1016/j.dam.2016.02.012

2. A graph in which $I^2[Ct(G)] \neq V(G)$

In this section first we present some results which led me to the construction of a graph G in which $I^2[Ct(G)] \neq V(G)$, but can be interesting on their own. Next, we present a graph for which $gin(Ct(G)) = 3$, that is $I[Ct(G)]$ is not geodetic.

In what follows G will be a connected graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A sequence of distinct vertices of G, $v_0, \ldots, v_k$, where $k \geq 0$, and v_i and v_{i+1}, $0 \leq i < k$, are adjacent, is a v_0–v_k *path* of length k. We will denote by $V(p)$ the set of vertices in a path p. A u–v *geodesic* is a u–v path of minimum length. The *distance* $d(u, v)$ between two vertices of G is the length of a u–v geodesic.

A vertex v is an *eccentric* vertex of u if $d(u, v) = ecc(u)$. The set of eccentric vertices of u is denoted as $\mathcal{E}(u)$. The maximum eccentricity among the vertices of G is the *diameter* of a graph G, denoted as $diam(G)$ and the minimum eccentricity is the *radius* of the graph, denoted as $rad(G)$. Given a vertex $v \in V(G)$, let us denote as $hi(v)$, the set of vertices such that $u \in hi(v)$ if and only if there exists a path $p = u_0, \ldots, u_k$, with $v = u_0$ and $u = u_k$, such that $ecc(u_{i+1}) = ecc(u_i) + 1$, $0 \leq i < k$ and $k > 0$; p will be called a *trail* from v to u. Note that if $v \in Ct(G)$ then $hi(v) = \emptyset$ and if $hi(v) \neq \emptyset$ then $hi(v) \cap Ct(G) \neq \emptyset$.

Lemma 1 ([2]). *Let G be a graph and let $v \in V(G)$, $u \in hi(v)$ and p a trail from v to u. For every eccentric vertex x of u there is a u–x geodesic containing v. Furthermore x is an eccentric vertex for each $z \in V(p)$.*

Lemma 2. *If $I[Ct(G)]$ is not geodetic then there exist six vertices a_i, $1 \leq i \leq 6$ such that*

1. $a_1 \notin I[I[Ct(G)]]$ *and* $a_1 \notin I[Ct(G)]$
2. $a_2 \in hi(a_1) \cap Ct(G)$
3. $a_3 \in \mathcal{E}(a_2)$, $a_3 \notin Ct(G)$
4. $a_4 \in hi(a_3) \cap Ct(G)$
5. $a_5 \in \mathcal{E}(a_4)$, $a_5 \notin Ct(G)$
6. $a_6 \in hi(a_5) \cap Ct(G)$
7. *The six vertices are distinct and* $ecc(a_6) \geq ecc(a_1) + 3$.

Proof of 1. If $I[Ct(G)]$ is not geodetic then let $a_1 \in V(G) \setminus I[I[Ct(G)]]$. Of course $a_1 \notin I[Ct(G)]$.

Proof of 2. Since by (1), $a_1 \notin Ct(G)$ then $hi(a_1) \neq \emptyset$. So we let a_2 be a vertex in $hi(a_1) \cap Ct(G)$.

Proof of 3. By Lemma 1, for every $v \in \mathcal{E}(a_2)$ there exists a a_2–v geodesic containing a_1. Let $a_3 \in \mathcal{E}(a_2)$. If $a_3 \in Ct(G)$ then $a_1 \in I[Ct(G)]$ a contradiction of (1).

Proof of 4. Let $a_4 \in hi(a_3) \cap Ct(G)$.

Proof of 5. If $I[Ct(G)]$ is not geodetic then, by Lemma 1, for every $v \in \mathcal{E}(a_4)$ we have that $v \notin Ct(G)$, for otherwise $a_3 \in I[Ct(G)]$ and then $a_1 \in I[I[Ct(G)]]$. Therefore let $a_5 \in \mathcal{E}(a_4)$.

Proof of 6. Let $a_6 \in hi(a_5) \cap Ct(G)$.

Proof of 7. Since $a_1 \notin Ct(G)$ then $ecc(a_2) > ecc(a_1)$. Therefore $a_2 \neq a_1$. It cannot be that $ecc(a_2) = 1$ for otherwise $ecc(a_1) = 0$ which is not possible. Therefore we have that $a_3 \notin \{a_2, a_1\}$. Furthermore since $ecc(a_4) > ecc(a_3) \geq ecc(a_2) > ecc(a_1)$ we have that $a_4 \notin \{a_3, a_2, a_1\}$. Analogously since $ecc(a_5) \geq ecc(a_4)$ we have that $a_5 \notin \{a_4, a_3, a_2, a_1\}$. Finally since $a_5 \notin Ct(G)$ then $ecc(a_6) > ecc(a_5)$ and $a_6 \notin \{a_5, a_4, a_3, a_2, a_1\}$. So all the six vertices are distinct. By what said above we have that

$$ecc(a_6) > ecc(a_5) \geq ecc(a_4) > ecc(a_3) \geq ecc(a_2) > ecc(a_1)$$

and then $ecc(a_6) \geq ecc(a_1) + 3$.

Theorem 3. *Let G be a graph. If $diam(G) \leq 6$, then $I[Ct(G)]$ is geodetic.*

Proof. In [1], it is shown that if $diam(G) \leq 4$, then $Ct(G)$ is geodetic. Suppose by contradiction that $5 \leq diam(G) \leq 6$ and that $I[Ct(G)]$ is not geodetic. Let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of six vertices as in Lemma 2. Since $diam(G) \leq 2\ rad(G)$ we have that $ecc(a_1) \geq 3$ and by (7) of Lemma 2, we would have that $ecc(a_6) \geq ecc(a_1)+3 \geq 6$ so we cannot have that $diam(G) = 5$. Suppose then that $diam(G) = 6$. If $ecc(a_1) > 3$, by (7) of Lemma 2, we would have that $ecc(a_6) \geq ecc(a_1) + 3 \geq 7$. So we must have that $ecc(a_1) = 3$. Let $v \in \mathcal{E}(a_6)$. Then $d(a_1, v) \leq 3$ and $d(a_1, a_6) \leq 3$. If $d(a_1, v) < 3$ or $d(a_1, a_6) < 3$ we can construct a path from a_6 to v whose length l is less than 6 (a contradiction). So we must have that $d(a_1, v) = d(a_1, a_6) = 3$. But then a_1 would be in a a_6–v geodesic where both a_6 and v are in $Ct(G)$, contrary to the hypothesis. □

Lemma 4. *Let G be a graph such that $I[Ct(G)]$ is not geodetic and $diam(G) = 7$. Let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of six vertices as in Lemma 2. Let p be a a_2–a_3 geodesic containing a_1 and let $a \in V(p)$ be the vertex adjacent to a_1 which is nearest to a_3. Then $ecc(a) \leq ecc(a_1)$.*