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Robust similarity between hypergraphs based on valuations and mathematical morphology operators

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ABSTRACT

This article aims at connecting concepts of similarity, hypergraph and mathematical morphology. We introduce new measures of similarity and study their relations with pseudo-metrics defined on lattices. More precisely, based on various lattices that can be defined on hypergraphs, we propose some similarity measures between hypergraphs based on valuations and mathematical morphology operators. We also detail new examples of these operators. The proposed similarity measures can be used in particular to introduce some robustness, up to some morphological operators. Some examples based on various dilations, erosions, openings and closings on hypergraphs illustrate the relevance of our approach. Potential applications to image comparison are suggested as well.

1. Introduction

In the field of automatic information processing, two aspects have gained importance in recent years. One consists in representing the structure of information, which is particularly crucial when dealing with large data, and the other is its increasing relation to algebraic processing methods relying on lattices.

The recent theory of hypergraphs takes its origin from the combinatorial set theory. First developed as a stand-alone mathematical theory, hypergraphs have become indispensable in many disciplines such as chemistry, physics, genetics, computer science, psychology… [30]. After having modeled data, an important task is the comparison of these data. Hence, the concept of similarity becomes very important. For instance, most of the disciplines cited above require the notions of comparison and similarity measures.

The notion of similarity plays a very important role in various fields of applied sciences. Classification is an example [8], and other examples such as indexing, retrieval or matching demonstrate the usefulness of the concept of similarity [9], with typical applications in image processing and image understanding. A recent trend in these domains is to rely on structural representations of the information (images for instance). Beyond the classical graph representations, and the associated notion of graph similarity, hypergraphs (in which edges can have any cardinality and are then called hyperedges), introduced in the 1960s [30], have recently proved useful. This concept has developed rapidly and has become both a powerful and well-structured mathematical theory for modeling complex situations. Let us consider the example of applications in image processing and understanding. Hypergraphs can be used to represent the structure of the image in different ways: vertices

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can be pixels and a hyperedge can be defined as a set of pixels sharing some properties, such as closeness in space or in a feature domain (color, texture…); it can also be defined at regional level, for representing spatial relations between regions or objects (which are then the vertices of the hypergraph) with arity greater than 2 (e.g. the "between" relation). In image applications, most similarity measures rely on features computed locally, or among the vertices of a hyperedge, and therefore do not completely exploit the structure of the hypergraph at this level [13,14,18,19].

Mathematical morphology offers interesting features to this purpose. Moreover, it has also strong algebraic bases, relying on the lattice theory [6,15,16,23–26].

In this paper we consider hypergraph representations of data, which can be endowed with a lattice structure, and mathematical morphology for manipulating these representations. Extending our preliminary work in [5], we propose new tools for defining similarity measures and metrics based on mathematical morphology. In order to deal with structured information, mathematical morphology has been developed on graphs [11,10,22,28,29], triangulated meshes [20], and more recently on simplicial complexes [12] and hypergraphs [3,4], where preliminary notions of dilation-based similarity were introduced. We propose to study similarities on lattices and more specifically on lattices of hypergraphs. We define some of them based on valuations on hypergraphs and mathematical morphology operators. They are illustrated on various types of lattices of hypergraphs. We also introduce new morphological operators, and show the interest of the proposed definitions for achieving robustness with respect to small variations of the compared hypergraphs. This robustness is intended in terms of indistinguishability, defined from morphological operators (i.e. two elements of a lattice are considered as indistinguishable if their images by a morphological operator are identical). Note that the paper is mainly theoretical, and although a few suggestions are provided to apply the proposed framework to images, such applications are out of the scope of the paper.

This paper is organized as follows. In Section 2 we recall some definitions on hypergraphs and lattices of hypergraphs on which morphological operators are defined. In Section 3, we show some general results on similarities, valuations and pseudo-metrics. Similarity and pseudo-metrics based on mathematical morphology are then defined in Section 4, with a number of illustrative examples.

2. Background and notations

Basic concepts on hypergraphs [1,7]: a *hypergraph H* denoted by $H = (V, E = (e_i)_{i\in I})$ is defined as a pair of a finite set V (vertices) and a family $(e_i)_{i\in I}$, (where I is a finite set of indices) of *hyperedges*. Hyperedges can be considered equivalently as subsets of vertices or as a relation between vertices of V. Let $(e_j)_{j\in\{1,2,\dots,l\}}$ be a sub-family of hyperedges of E. The set of vertices belonging to these hyperedges is denoted by $v(\cup_{j\in\{1,2,\dots,l\}} e_j)$, and $v(e)$ denotes the set of vertices forming the hyperedge e (note that $v(e) = e$ if hyperedges are considered as subsets of vertices). If $\cup_{i\in I} v(e_i) = V$, the hypergraph is without *isolated vertex* (a vertex x is isolated if $x \in V \setminus \cup_{i\in I} v(e_i)$). The set of isolated vertices is denoted by $V_{\setminus E}$. By definition the *empty hypergraph* is the hypergraph $H_\emptyset$ such that $V = \emptyset$ and $E = \emptyset$. We denote by $H(x)$ the star centered at x, for $x \in V$, i.e. the set of hyperedges containing x. A hypergraph is called *simple* if $\forall (i, j) \in I^2,\ v(e_i) \subseteq v(e_j) \Rightarrow i = j$. The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$ (where $\sqcup$ stands for the disjoint union), and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in v(e)$. Edges of a bipartite graph are considered as non directed. Conversely, to each bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$, we can associate two hypergraphs: a hypergraph $H = (V, E)$, where $V = V_1$ and E contains the neighborhood of any element of V_2, where the neighborhood is considered in the bipartite graph Γ (i.e. $E = \{N_\Gamma(x) \mid x \in V_2\}$, where $N_\Gamma(x)$ is the neighborhood of a vertex in the graph Γ, derived from A), and its dual $H^* = (V^*, E^*)$, where $V^* = V_2$ and E^* is defined similarly.

Mathematical morphology on hypergraphs: in [4], we introduced mathematical morphology on hypergraphs. The first step was to define complete lattices on hypergraphs. Then the whole algebraic apparatus of mathematical morphology applies [6,15,16,24,26].

Let $(\mathcal{T}, \preceq)$ and $(\mathcal{T}', \preceq')$ be two complete lattices. All the following definitions and results are common to the general algebraic framework of mathematical morphology in complete lattices [6,15,16,24,26].

Definition 1. An operator $\delta : \mathcal{T} \to \mathcal{T}'$ is a dilation if: $\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)$, where $\vee$ denotes the supremum associated with $\preceq$ and $\vee'$ the one associated with $\preceq'$.

An operator $\varepsilon : \mathcal{T}' \to \mathcal{T}$ is an erosion if: $\forall (x_i) \in \mathcal{T}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)$, where $\wedge$ and $\wedge'$ denote the infimum associated with $\preceq$ and $\preceq'$, respectively.

An operator ψ on $\mathcal{T}$ is a morphological filter if it is increasing and idempotent. An anti-extensive filter is an algebraic opening and an extensive filter is an algebraic closing.

We denote the universe of hypergraphs by $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices (that we assume to be finite) and $\mathcal{E}$ the set of hyperedges. The powersets of $\mathcal{V}$ and $\mathcal{E}$ are denoted by $\mathcal{P}(\mathcal{V})$ and $\mathcal{P}(\mathcal{E})$, respectively. We denote a hypergraph by $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$. As developed in [4], several complete lattices can be built on $(\mathcal{V}, \mathcal{E})$. Let us denote by $(\mathcal{T}, \preceq)$ any of these lattices. We denote by $\wedge$ and $\vee$ the infimum and the supremum, respectively. The least element is denoted by $0_\mathcal{T}$ and the greatest element by $1_\mathcal{T}$. Here we will use three examples of complete lattices:

1. $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ (lattice over the power set of vertices);
2. $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$ (lattice over the power set of hyperedges);