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Note

On the construction of non-2-colorable uniform hypergraphs

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ABSTRACT

The problem of 2-coloring uniform hypergraphs has been extensively studied over the last few decades. An n-uniform hypergraph is not 2-colorable if its vertices cannot be colored with two colors, Red and Blue, such that every hyperedge contains Red as well as Blue vertices. The least possible number of hyperedges in an n-uniform hypergraph which is not 2-colorable is denoted by $m(n)$. In this paper, we consider the problem of finding an upper bound on $m(n)$ for small values of n. We provide constructions which improve the existing results for some such values of n. We obtain the first improvement in the case of $n = 8$.

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1. Introduction

A hypergraph is said to have property B if there exists a proper subset S of its vertices such that every hyperedge of the hypergraph contains vertices from both S and $\overline{S}$, the complement of S. In other words, if the vertices of a hypergraph can be colored using Red and Blue colors such that every hyperedge of the hypergraph contains Red as well as Blue vertices, then that hypergraph is said to have *property B*. It is known that there exist 2-colorable hypergraphs with arbitrarily high number of hyperedges. For example, Lovász mentions the following in Problem 13.33 of [10]: if there are no two hyperedges with exactly one common vertex in a hypergraph, then the hypergraph is 2-colorable.

The least possible number of hyperedges in an n-uniform hypergraph which does not have property B is denoted by $m(n)$. One motivation to study such hypergraphs is the following relationship between non-2-colorable hypergraphs and unsatisfiable CNF formulas. In fact, constructing a non-2-colorable n-uniform hypergraph H with x hyperedges is equivalent to constructing an unsatisfiable monotone n-CNF with $2x$ clauses. For a given n-uniform hypergraph H, let H' denote the n-CNF obtained by adding clauses $C_e := (x_1 \vee x_2 \ldots \vee x_n)$ and $\bar{C}_e := (\bar{x}_1 \vee \bar{x}_2 \ldots \vee \bar{x}_n)$ for every hyperedge $e = \{x_1, x_2, \ldots, x_n\} \in H$. Note that H' is monotone, i.e., every clause of H' either contains only non-negated literals or only negated literals. It can be easily seen that every 2-coloring χ of H yields a satisfying assignment α of H' and vice versa.

A lot of work has been done to find a lower bound on $m(n)$. Erdős [4] showed that $m(n) = \Omega(2^n)$, which was improved to $\Omega(n^{1/3-o(1)}2^n)$ by Beck [3]. It was further improved by Radhakrishnan and Srinivasan [13] to the currently best known lower bound $m(n) = \Omega(\sqrt{\frac{n}{\ln n}}2^n)$. In the other direction, Erdős [5] used the probabilistic method to show that $m(n) = O(n^2 2^n)$. However, as it is typical in such constructions, this method did not provide any explicit construction that matches this bound. For a general n, only a few explicit constructions of non-2-colorable uniform hypergraphs are known. Recently, Gebauer [7] provided a construction which produces a non-2-colorable n-uniform hypergraph containing $2^n \cdot 2^{O(n^{\frac{2}{3}})}$ hyperedges for a sufficiently large n. This is the asymptotically best-known constructive upper bound on $m(n)$ till now. Some constructions

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for small values of n are mentioned in [8]. However, all the known constructions give an upper bound on $m(n)$ which is asymptotically far from Erdős's non-constructive upper bound.

Finding $m(n)$ for smaller values of n is also an extensively studied topic. It is straightforward to see that $m(2) = 3$ (triangle graph) and $m(3) = 7$ (Fano plane [9]). But, finding $m(n)$ for $n \geq 4$ remained an open problem for a long time. The construction of Abbott and Hanson [1] proved that $m(4) \leq 24$. Seymour [14] improved upon this construction to show that $m(4) \leq 23$. In the other direction, Manning [11] provided a proof that $m(4) \geq 21$. Recently, Östergård [12] has proved that $m(4) \geq 23$. This shows that $m(4) = 23$, and settles this problem for $n = 4$. The problem of finding the exact value of $m(n)$ for $n \geq 5$ still remains an open problem.

The best known upper bound on $m(n)$ for small values of n are obtained from three constructions where each construction provides a recurrence relation. We describe these constructions below.

The first construction is due to Abbott and Moser [2] for all composite values of n. They proved that if $n = a \cdot b$ such that a and b are integer factors of n, then $m(n) \leq m(a) \cdot (m(b))^a$. Let us discuss their construction in brief. Take an a-uniform hypergraph giving the best known value for $m(a)$ and replace the vertices of this hypergraph with different b-uniform hypergraphs that gives the best known value for $m(b)$. Note that the vertex set of each of these b-uniform hypergraphs must be disjoint from each other. In order to form an n-uniform hyperedge, the hyperedges of these a different b-uniform hypergraphs are combined with each other in every possible way. Therefore, the n-uniform hypergraph has a total of $m(a) \cdot (m(b))^a$ hyperedges. A more detailed description of this construction and the proof for the claim that the hypergraph formed by this construction is not 2-colorable can be found in [2].

The second construction is by Abbott and Hanson [1] which gives the recurrence $m(n) \leq 2^{n-1} + n \cdot m(n-2)$ when n is odd. The third construction is by Toft [15], who obtained the following bound when n is even: $m(n) \leq 2^{n-1} + \binom{n}{n/2}/2 + n \cdot m(n-2)$. In Section 2, we discuss these two constructions in some details. Exoo [6] tried to improve the upper bounds on $m(n)$ for small values of n using computer programs. However, his constructions did not improve any of the known upper bounds obtained from the recurrences mentioned above [8].

1.1. Our contribution

In Section 3, we provide a construction that improves the currently best known upper bounds on $m(n)$ for some small values of n, starting from $n = 13$.

Result 1. *We provide a construction which shows that* $m(n) \leq (n+1) \cdot 2^{n-2} + (n-1) \cdot m(n-2)$ *when n is odd, and* $m(n) \leq (n+1) \cdot 2^{n-2} + \binom{n}{n/2}/2 + (n-1) \cdot \left(m(n-2) + \binom{n-2}{(n-2)/2}\right)$ *when n is even.*

However, in Section 4, we provide another construction that improves the construction given in Section 3.

Result 2. *We provide a construction which shows that* $m(n) \leq (n+4) \cdot 2^{n-3} + (n-2) \cdot m(n-2)$ *when n is odd, and* $m(n) \leq (n+4) \cdot 2^{n-3} + n \cdot \binom{n-2}{(n-2)/2}/2 + \binom{n}{n/2}/2 + (n-2) \cdot m(n-2)$ *when n is even.*

Using this recurrence relation, the first improvement is obtained when $n = 11$. We show that $m(11) \leq 25{,}449$, which improves the currently best known bound $m(11) \leq 27{,}435$.

In Section 5, we provide a construction for $n = 8$ that improves the currently best known bound $m(8) \leq 1339$.

Result 3. $m(8) \leq 1269$.

2. The construction of Abbott and Hanson [1], and Toft [15]

In this section, we discuss the construction made by Abbott and Hanson [1], which was further improved upon by Toft [15]. In fact, Abbott and Hanson's [1] construction is good for odd values of n, while Toft [15] improved their construction for even values of n. Now let us discuss their construction, which we call as the *AHT construction*, in detail.[1]

For a given $n \geq 3$, let us consider an $(n-2)$-uniform hypergraph producing the best known upper bound for $m(n-2)$. We denote this hypergraph as a *core hypergraph*. Let m_{n-2} be the number of hyperedges present in this core hypergraph $C = (X, Y)$. The set of hyperedges is denoted by $Y = \{e_1, e_2, e_3, \ldots, e_{m_{n-2}}\}$, where e_i is the ith hyperedge in the core hypergraph C. Let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and $V = \{v_1, v_2, v_3, \ldots, v_n\}$ denote two disjoint set of vertices, each of them disjoint from X, the set of vertices in the core hypergraph C. For each $1 \leq i \leq n$, we call u_i and v_i to be a pair of *matching vertices*. For any $K = \{a_1, a_2, \ldots, a_k\}$ which is a proper subset of $\{1, 2, \ldots, n\}$ such that $1 \leq a_1 < a_2 \ldots < a_k \leq n$, we denote $U_K = \{u_{a_1}, u_{a_2}, \ldots, u_{a_k}\}$ and $V_K = \{v_{a_1}, v_{a_2}, \ldots, v_{a_k}\}$. We also define $\bar{U}_K = U \setminus U_K$ and $\bar{V}_K = V \setminus V_K$.

[1] Note that the variables used in a section is valid only inside that section.

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