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Hull number: P_5-free graphs and reduction rules[☆]

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ABSTRACT

In this paper, we study the (geodesic) hull number of graphs. For any two vertices $u, v \in V$ of a connected undirected graph $G = (V, E)$, the closed interval $I[u, v]$ of u and v is the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is (geodesically) convex if $I[S] = S$. Given a subset $S \subseteq V$, the convex hull $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a hull set of G if $I_h[S] = V$. The size of a minimum hull set of G is the hull number of G, denoted by $hn(G)$.

First, we show a polynomial-time algorithm to compute the hull number of any P_5-free triangle-free graph. Then, we present four reduction rules based on vertices with the same neighborhood. We use these reduction rules to propose a fixed parameter tractable algorithm to compute the hull number of any graph G, where the parameter can be the size of a vertex cover of G or, more generally, its neighborhood diversity, and we also use these reductions to characterize the hull number of the lexicographic product of any two graphs.

1. Introduction

All graphs in this work are undirected, simple and loop-less. Given a connected graph $G = (V, E)$, the closed interval $I[u, v]$ of any two vertices $u, v \in V$ is the set of vertices that belong to some u–v geodesic of G, i.e. some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *(geodesically) convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that a vertex v is *generated* by a set of vertices S if $v \in I_h[S]$. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$ [8].

It is known that the problem of computing $hn(G)$ is NP-hard for bipartite graphs [3]. Several bounds on the hull number of triangle-free graphs are presented in [7]. In [6], the authors show, among other results, that the hull number of any P_4-free graph, i.e. any graph without induced path with four vertices, can be computed in polynomial time. In Section 3, we show a linear-time algorithm to compute the hull number of any P_5-free triangle-free graph.

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In Section 4, we show four reduction rules to obtain, from a graph G, another graph G^* that has one vertex less than G and which satisfies either $hn(G) = hn(G^*)$ or $hn(G) = hn(G^*) + 1$, according to the used rule. We then first use these rules to obtain a fixed parameter tractable (FPT) algorithm, where the parameter is the neighborhood diversity of the input graph. For definitions on Parameterized Complexity we refer to [9]. Given a graph G and vertices $u, v \in V(G)$, we say that u and v are *twins* (a.k.a. of the *same type*) if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. The *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are twins. This parameter was proposed in [11], motivated by the fact that a graph of bounded vertex cover also has bounded neighborhood diversity, and therefore the later parameter can be used to obtain more general results. To see that a graph of bounded vertex cover has bounded neighborhood diversity, let G be a graph that has a minimal vertex cover $S \subseteq V(G)$ of size k, and let $I = V(G) \setminus S$. Since S is a vertex cover, I is an independent set. Therefore, the neighborhood of any vertex in I is contained in S, and since there are 2^k distinct subsets of S, there exists a partition by vertices of the same type of I with at most 2^k parts. The vertices in S may be partitioned in k sets of singletons, what gives a partition of the vertices of the graph into $k + 2^k$ sets of twin vertices. Then, the neighborhood diversity of the graph is at most $k + 2^k$. Many problems have been shown to be FPT when the parameter is the neighborhood diversity [10].

Finally, we use these rules to characterize the hull number of the lexicographic product of any two graphs. Given two graphs G and H, the *lexicographic product* $G \circ H$ is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1g_2 \in E(G)$ or we have that both $g_1 = g_2$ and $h_1h_2 \in E(G)$.

It is known in the literature a characterization of the (geodesic) convex sets in the lexicographic product of two graphs [1] and a study of the pre-hull number for this product [13]. There are also some results concerning the hull number of the Cartesian and strong products of graphs [4,5].

2. Preliminaries

Let us recall some definitions and lemmas that we use in the sequel.

We denote by $N_G(v)$ (or simply $N(v)$) the neighborhood of a vertex. A vertex v is *simplicial* (resp. *universal*) if $N(v)$ is a clique (resp. is equal to $V(G) \setminus \{v\}$). Let $d_G(u, v)$ denote the *distance* between u and v, i.e. the length of a shortest (u, v)-path. A subgraph $H \subseteq G$ is *isometric* if, for each $u, v \in V(H)$, $d_H(u, v) = d_G(u, v)$. A P_k (resp. C_k) in a graph G denotes an induced path (resp. cycle) on k vertices. Given a graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph. Moreover, we consider that all the graphs in this work are connected. Indeed, if a graph G is not connected, its hull number can be computed by the sum of the hull numbers of its connected components [6].

Lemma 1 ([8]). *For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2 ([6]). *Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3 ([6]). *Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4 ([6]). *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \setminus S$ is convex, then every hull set of G contains at least one vertex of S.*

3. Hull number of P_5-free triangle-free graphs

In this section, we present a linear-time algorithm to compute $hn(G)$, for any P_5-free triangle-free graph G. To prove the correctness of this algorithm, we need to recall some definitions and previous results:

Definition 1. Given a graph $G = (V, E)$, we say that $S \subseteq V$ is a dominating set if every vertex $v \in V \setminus S$ has a neighbor in S.

It is well known that:

Theorem 1 ([12]). *G is a connected P_5-free graph if, and only if, for every induced subgraph $H \subseteq G$ either $V(H)$ contains a dominating C_5 or a dominating clique.*

As a consequence, we have that:

Corollary 1. *If G is a connected P_5-free bipartite graph, then there exists a dominating edge in G.*

Theorem 2. *The hull number of a P_5-free bipartite graph $G = (A \cup B, E)$ can be computed in linear time.*

Proof. By Corollary 1, G has at least one dominating edge. Observe that the dominating edges of a bipartite graph can be found in linear time by computing the degree of each vertex and then considering the sum of the degrees of the endpoints of each edge. For a dominating edge, this sum is equal to the number of vertices.