Discrete Applied Mathematics 205 (2016) 126–131

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# On the roots of domination polynomial of graphs

Mohammad Reza Oboudi *

*Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran*
*School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

ARTICLE INFO

*Article history:*
Received 1 January 2015
Received in revised form 17 October 2015
Accepted 1 December 2015
Available online 11 January 2016

*Keywords:*
Graph
Dominating set
Domination polynomial
Domination root

ABSTRACT

Let $G$ be a graph of order $n$. A dominating set of $G$ is a subset of vertices of $G$, say $S$, such that every vertex in $V(G)\setminus S$ is adjacent to at least one vertex of $S$. The domination polynomial of $G$ is the polynomial $D(G,x)=\sum_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of $G$ of size $i$. A root of $D(G,x)$ is called a domination root of $G$. Let $\delta=\delta(G)$ be the minimum degree of vertices of $G$. We prove that all roots of $D(G,x)$ lies in the set $\{z : |z+1| \leq \sqrt[\delta+1]{2^n-1}\}$. We show that $D(G,x)$ has at least $\delta-1$ non-real roots. In particular, we prove that if all roots of $D(G,x)$ are real, then $\delta=1$. We construct an infinite family of graphs such that all roots of their polynomials are real. Motivated by a conjecture (Akbari, et al. 2010) which states that every integer root of $D(G,x)$ is $-2$ or 0, we prove that if $\delta \geq \frac{2n}{3}-1$, then every integer root of $D(G,x)$ is $-2$ or 0. Also we prove that the conjecture is valid for trees and unicyclic graphs. Finally we characterize all graphs that their domination roots are integer.



## 1. Introduction

Throughout this paper we will consider only simple graphs. Let $G=(V,E)$ be a graph. The *order* of $G$ denotes the number of vertices of $G$. For two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, the *disjoint union* of $G_1$ and $G_2$ denoted by $G_1+G_2$ is the graph with vertex set $V_1\cup V_2$ and edge set $E_1\cup E_2$. The graph $rG$ denotes the disjoint union of $r$ copies of $G$. For every vertex $v\in V$, the *open neighborhood* of $v$ is the set $N(v)=\{u\in V : uv\in E\}$ and the *closed neighborhood* is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$, the open neighborhood is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood is $N[S]=N(S)\cup S$. A set $S\subseteq V$ is a *dominating set* of $G$ if $N[S]=V$, or equivalently, every vertex in $V\setminus S$ is adjacent to at least one vertex of $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. For a detailed treatment of this parameter, the reader is referred to [16]. We denote the complete graph of order $n$, the complete bipartite graph with part sizes $m,n$, the cycle of order $n$, and the path of order $n$, by $K_n$, $K_{m,n}$, $C_n$, and $P_n$, respectively. Also $K_{1,n}$ is called a *star*. For every vertex $v\in V(G)$, the *degree* of $v$ is the number of edges incident with $v$. By $\delta(G)$ and $\Delta(G)$ we mean the minimum and maximum degree of vertices of $G$, respectively.

Let $\mathcal{D}(G,i)$ be the family of all dominating sets of a graph $G$ with cardinality $i$ and let $d(G,i)=|\mathcal{D}(G,i)|$. The *domination polynomial* $D(G,x)$ of $G$ was defined by Alikhani and Peng, see [3–5], as $D(G,x)=\sum_{i=1}^{|V|} d(G,i)x^i$. For example the domination polynomial of the complete graph on $n$ vertices, $K_n$, is $D(K_n,x)=(x+1)^n-1$. Every root of $D(G,x)$ is called a *domination root* of $G$. We denote the set of all roots of $D(G,x)$ by $Z(D(G,x))$. We note that $d(G,n)=1$ and the degree of the polynomial $D(G,x)$ is $n$. Since all coefficients of $D(G,x)$ are positive, every real root of $D(G,x)$ is non-positive. Also zero is a root of $D(G,x)$ for every graph $G$.

* Correspondence to: Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran.
*E-mail addresses:* mr_oboudi@shirazu.ac.ir, mr_oboudi@yahoo.com.

http://dx.doi.org/10.1016/j.dam.2015.12.010

Let $A \subseteq B \subseteq V$. Define $\mathcal{D}_{A,B}(G, i)$ as follows:

$$\mathcal{D}_{A,B}(G, i) = \{S \in \mathcal{D}(G, i) : S \cap B = A\}.$$

Let $d_{A,B}(G, i) = |\mathcal{D}_{A,B}(G, i)|$ and define $D_{A,B}(G, x) = \sum_{i=1}^{|V|} d_{A,B}(G, i)x^i$ [1].

The *corona* of two graphs $G_1$ and $G_2$, as defined by Frucht and Harary in [14], is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the ith vertex of $G_1$ is adjacent to every vertex in the ith copy of $G_2$. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added.

The roots of graph polynomials reflect some important information about the structure of graphs. There are many papers on the location of the roots of graph polynomials such as chromatic polynomial, matching polynomial, independence polynomial and characteristic polynomial [21]. In [24] Thomassen showed that the chromatic polynomial has no root in the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \frac{32}{27}]$. Moreover he proved that the roots of the chromatic polynomials are dense in the interval $[\frac{32}{27}, \infty)$. He also showed that if the chromatic polynomial of a graph has a non-integer root less than or equal to $1.29559\ldots$, then the graph has no Hamiltonian path [25]. In [9] Brown, Hickman, and Nowakowski proved that the real roots of the independence polynomials are dense in the interval $(-\infty, 0]$, while the complex roots are dense in the complex plane. There are many results on the roots of the matching polynomials as well. In [17] it was proved that all roots of the matching polynomials are real. Also it was shown that if a graph has a Hamiltonian path, then all roots of its matching polynomial are simple (see Theorem 4.5 of [15]). It is well known that all roots of the characteristic polynomials are real. For more details on the characteristic polynomials see [12]. There are also many bounds for the roots of these polynomials in terms of other parameters of the graphs. For instance, in [23] Sokal proved that for every graph $G$, the absolute value of any root of the chromatic polynomial of $G$ is at most $8\Delta(G)$ ($\Delta(G)$ is the maximum degree of $G$).

In [11] it was proved that all roots of edge cover polynomials lie in the following set:

$$\left\{ z \in \mathbb{C} : |z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}} \simeq 5.099 \right\}.$$

Averbouch, Godlin, and Makowsky [7] introduced a new graph polynomial, the *edge elimination polynomial* that is denoted by $\xi(G, x, y, z)$, for every graph $G$, which generalizes some well known graph polynomials such as chromatic polynomials, matching polynomials, independence polynomials, Tutte polynomials and edge cover polynomials.

In this paper we study the location of domination roots. Related to the roots of domination polynomials there are a few papers. See [1,2,10] for more details. Recently in [10] Brown and Tufts studied the location of the roots of domination polynomials for some families of graphs such as bipartite cocktail party graphs and complete bipartite graphs. In particular, they showed that the set of all domination roots is dense in the complex plane.

The structure of this paper is the following. In the next section, we state a formula for computing the domination polynomials of graphs and introduce an upper bound for the absolute value of the domination roots. Also we study the real domination roots and those graphs such that all domination roots are real. We construct an infinite family of graphs in which all domination roots are real. Finally in the last section we investigate integer domination roots and characterize all graphs that all their domination roots are integer.

## 2. Locations of the roots of domination polynomials

In this section we investigate about the location of the roots of domination polynomial of graphs. First we introduce a formula to compute the domination polynomial of graphs. This formula also has been proved in [13] by a different method.

**Lemma 1.** *Let $G$ be a graph of order $n$. Then $D(G, x) = \sum_{S \subseteq V(G)} (-1)^{|S|}(x + 1)^{n-|N[S]|}$.*

**Proof.** We note that $(x+1)^n$ is the generating function of the sequence of the number of subsets of $V(G)$. On the other hand for any $S \subseteq V(G)$, $(x + 1)^{n-|N[S]|}$ is the generating function of the sequence of the subsets of $V(G)$ which has no neighbor in $S$. Now the desired statement follows from the inclusion–exclusion principle. □

As a consequence of Lemma 1 we indicate the location of the roots of domination polynomials.

**Theorem 1.** *Let $G$ be a graph of order $n$. Let $\delta = \delta(G)$. Then all roots of $D(G, x)$ lie in the circle with center $(-1, 0)$ and the radius $\sqrt[\delta+1]{2^n - 1}$; that is, if $D(G, z) = 0$, then $|z + 1| \leq \sqrt[\delta+1]{2^n - 1}$.*

**Proof.** Let $y = z + 1$. Using Lemma 1 one has $D(G, z) = \sum_{S \subseteq V(G)} (-1)^{|S|} y^{n-|N[S]|}$. Let $f(y) = \sum_{\emptyset \neq S \subseteq V(G)} (-1)^{|S|} y^{n-|N[S]|}$. We can rewrite $D(G, z) = y^n - f(y)$. If $S$ is a non-empty subset of $V(G)$, then $|N[S]| \geq \delta + 1$. Suppose that $|y| \geq 1$. Then $|y|^{n-|N[S]|} \leq |y|^{n-\delta-1}$, for every $\emptyset \neq S \subseteq V(G)$. Therefore $|f(y)| \leq (2^n - 1)|y|^{n-\delta-1}$.

Now, let $|y| > \sqrt[\delta+1]{2^n - 1}$. Then $|y| \geq 1$. So $|f(y)| \leq (2^n - 1)|y|^{n-\delta-1} < |y|^n$. This shows that $|y^n - f(y)| > 0$. Equivalently $|D(G, z)| > 0$. This completes the proof. □