Discrete Applied Mathematics 202 (2016) 197–201

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

A note on the computation of the fraction of smallest denominator in between two irreducible fractions

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A R T I C L E I N F O

Article history:
Received 20 April 2015
Received in revised form 25 August 2015
Accepted 27 August 2015
Available online 19 September 2015

Keywords:
Simplest fraction
Continued fractions
Algorithm

A B S T R A C T

Given two irreducible fractions f and g, with $f < g$, we characterize the fraction h such that $f < h < g$ and the denominator of h is as small as possible. An output-sensitive algorithm of time complexity $\mathcal{O}(d)$, where d is the depth of h is derived from this characterization.

1. Introduction

Given two irreducible fractions defining an interval, we investigate the problem of finding the fraction of smallest denominator in this interval. Keeping integers as small as possible can be crucial for exact arithmetic computations. Indeed, even if exact arithmetic libraries [3] provide efficient tools to handle exact computations on arbitrarily large integers, dealing with big integers remains computationally expensive. Thus, when a parameter value has to be chosen in a given interval, it may be interesting to choose the value involving integers as small as possible to enable fast computations. This problem also has a nice geometric interpretation and arises for instance in the computation of the integer hull of a polygon [5].

In this note, we consider two irreducible proper fractions f and g such that $0 \leq f < g \leq 1$. We give a characterization of the irreducible proper fraction h such that $f < h < g$ and the denominator of h is as small as possible. Finally, we provide an output-sensitive algorithm to compute h from the continued fraction decompositions of f and g.

2. Problem statement and theorem

A *simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

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http://dx.doi.org/10.1016/j.dam.2015.08.029
0166-218X/

The numbers a_i are called *partial quotients*. In the following, we restrict ourselves to the case where all a_i are positive integers. Any rational number f can be expressed as a simple continued fraction with a finite number of partial quotients. We denote $f = \frac{p}{q} = [a_0; a_1, a_2 \dots a_n], p, q \in \mathbb{N}$. The mth *principal convergent* of f is equal to $f_m = \frac{p_m}{q_m} = [a_0; a_1, a_2 \dots a_m]$. The sequence of fractions f_{2m}, $m \in 0 \dots \lfloor \frac{n}{2} \rfloor$, is increasing while the sequence of fractions f_{2m+1}, $m \in 0 \dots \lfloor \frac{n}{2} \rfloor$ is decreasing. Moreover, for any $a_i > 1$, the *intermediate convergents* are defined as $f_{i,k} = \frac{p_{ik}}{q_{ik}} = [a_0; a_1 \dots a_{i-1}, k]$ with $1 \le k < a_i$. When $k = a_i$, we have $f_{i,a_i} = f_i$. In the following, convergent stands for either a principal or an intermediate convergent.

For a given fraction f, we define two ordered sets of convergents as follows (see also [1, Chap. 32, Parag. 15]): $\Gamma_{even}(f) = \{f_{2m,k}, k \in 1 \dots a_{2m}, m \in 0 \dots \lfloor \frac{n}{2} \rfloor\}$, $\Gamma_{odd}(f) = \{f_{2m+1,k}, k \in 1 \dots a_{2m+1}, m \in 0 \dots \lfloor \frac{n-1}{2} \rfloor\}$. If $a_{2m} > 1$, we have $f_{2m,k-1} < f_{2m,k}$ for any $m \in 0 \dots \lfloor \frac{n}{2} \rfloor$ and any $k \in 2 \dots a_{2m}$, and if $n \ge 2$, $f_{2m,a_{2m}} < f_{2m+2,1}$ for any $m \in 0 \dots \lfloor \frac{n}{2} \rfloor - 1$. Similarly, if $a_{2m+1} > 1$ we have $f_{2m+1,k-1} > f_{2m+1,k}$ for any $m \in 0 \dots \lfloor \frac{n-1}{2} \rfloor$ and any $k \in 2 \dots a_{2m}$, and if $n \ge 3$, $f_{2m+1,a_{2m}} > f_{2m+3,1}$ for any $m \in 0 \dots \lfloor \frac{n-1}{2} \rfloor - 1$.

For the sake of clarity, we rename the elements of $\Gamma_{even}(f)$ and $\Gamma_{odd}(f)$ as follows: $\Gamma_{even}(f) = \{\gamma_0 = [a_0], \gamma_2 \dots \gamma_{2i} \dots\}$ and $\Gamma_{odd}(f) = \{\dots \gamma_{2i+1} \dots \gamma_3, \gamma_1 = [a_0; a_1]\}$.

Definition 1. A fraction f is said to be *less complex* (resp. *more complex*) than a fraction g if and only if the denominator of f is strictly lower (resp. strictly greater) than the denominator of g.

Before stating our main theorem, let us recall some classical definitions. The *Farey sequence of order* n is the ascending sequence of irreducible fractions between 0 and 1 whose denominator does not exceed n [4]. The *mediant fraction* of two given fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ is defined as $\frac{p+p'}{q+q'}$. Given three successive terms $\frac{p}{q}$, $\frac{p'}{q'}$ and $\frac{p''}{q''}$ of a Farey sequence, $\frac{p'}{q'}$ is the mediant of $\frac{p}{q}$ and $\frac{p''}{q''}$ and we have $p'q - pq' = 1$ (similarly, $p''q' - p'q'' = 1$).

Theorem 1. *Let f and g be two irreducible fractions such that $0 \le f < g \le 1$. Let $\mathcal{H} = \{\frac{p}{q} \mid f < \frac{p}{q} < g\}$, and $h_{\min} = \arg\min_{\frac{p}{q} \in \mathcal{H}} \{q\}$. Then,*

(i) *the fraction $h_{\min}$ is well (uniquely) defined;*
(ii) *if f and g are successive terms of a Farey sequence, $h_{\min}$ is equal to the mediant of f and g;*
(iii) *otherwise, if g is a convergent of f, $g = \gamma_i \in \Gamma_{odd}(f)$, then $h_{\min} = \gamma_{i+2}$;*
(iv) *otherwise, if f is a convergent of g, $f = \gamma_i \in \Gamma_{even}(g)$, then $h_{\min} = \gamma_{i+2}$;*
(v) *otherwise, $h_{\min}$ is equal to the unique fraction in $\mathcal{H}$ that is a convergent of both f and g.*

The proof uses the following lemmas.

Lemma 1 (*[1, Chap. 32, Parag. 15]*). *For any fraction f, "it is impossible between any consecutive pair of either" $\Gamma_{even}(f)$ or $\Gamma_{odd}(f)$ "to insert a fraction which shall be less complex than the more complex of the two".*

Lemma 2. *Let $\frac{p}{q}$ and $\frac{p'}{q}$ be two irreducible fractions with $0 < \frac{p}{q} < \frac{p'}{q} < 1$, $q \neq 1$. There exists a fraction $\frac{a}{b}$ such that $b < q$ and $\frac{p}{q} < \frac{a}{b} < \frac{p'}{q}$.*

Proof. Suppose that there is no fraction of denominator strictly lower than q between $\frac{p}{q}$ and $\frac{p'}{q}$. Then $\frac{p}{q}$ and $\frac{p'}{q}$ are two successive terms of the Farey sequence of order q. Therefore, $p'q - qp = 1$, and rewriting p' as $p + k$ with $k \in \mathbb{N}$, $k \ge 1$, we get $qk = 1$, which leads to a contradiction.

Lemma 3. *For any fraction f, and for all $i, j \in \mathbb{N}$, if γ_i and γ_j are two elements of $\Gamma_{even}(f)$ or $\Gamma_{odd}(f)$, then γ_i is more complex than γ_j if and only if $i > j$.*

Proof. The proof is straightforward from the definition of principal and intermediate convergents.

Proof of Theorem 1. (i) Suppose that there exist two fractions $h = \frac{a}{b}$ and $h' = \frac{a'}{b}$ such that $b = \min_{\frac{p}{q} \in \mathcal{H}} \{q\}$ and $h, h' \in \mathcal{H}$. From the Theorem hypothesis, $0 < h < 1$, and thus $b > 1$. Then Lemma 2 directly leads to a contradiction. Moreover, a similar argument shows that the denominator of $h_{\min}$ is different from both the denominators of f and g.

(ii) In this case, as stated in [1, Chap. 32, Parag. 12] for instance, any fraction lying between f and g is more complex than f and g. Consequently, h is the fraction such that f, h and g are successive terms of a Farey sequence of order strictly greater than $\max\{f_q, g_q\}$, and by definition, h is the mediant of f and g.

(iii) Let $h \in \mathcal{H}$, $h \neq \gamma_{i+2}$. We show that h is more complex than γ_{i+2}. First, if $\gamma_{i+2} < h < g$, then Lemma 1 enables to conclude. If there exists j such that $h = \gamma_j$, then $j > i + 2$ since $h \in \mathcal{H}$, and we conclude using Lemma 3. Finally, if there exists j such that $\gamma_{j+2} < h < \gamma_j$, $j > i + 2$ then from Lemma 3, γ_j is more complex than γ_{i+2} and from Lemma 1,