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On some identities for the Fibonomial coefficients via generating function

Pavel Trojovský*

Department of Mathematics, University Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czech Republic

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Abstract

Some new identities for the Fibonomial coefficients are derived. These identities are related to the generating function of the kth powers of the Fibonacci numbers. Proofs are based on manipulation with the generating function of the sequence of "signed Fibonomial triangle".

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1. Introduction

In 1915 Fontené in [1] suggested a generalization of binomial coefficients, replacing the natural numbers by the terms of any sequence $\{A_n\}$ of real or complex numbers. The sequence $\{A_n\}$ is essentially arbitrary but it is required that $A_0 = 0$ and $A_n \neq 0$ for $n \geqslant 1$. Hence he considered the generalized binomial coefficients defined by $\left\{ {n \atop k} \right\} = (A_n A_{n-1} \cdots A_{n-k+1}/A_1 A_2 \cdots A_k)$ and he gave the fundamental recurrence for them by

$$\left\{ {n \atop k} \right\} - \left\{ {n-1 \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} \frac{A_n - A_k}{A_{n-k}}.$$

Many papers were devoted to the generalized binomial coefficients for example see [2–4,7] and [12].

Since 1964, there has been an accelerated interest in Fibonomial coefficients, which correspond to the choice $A_n = F_n$, where F_n are the Fibonacci numbers defined by $F_{n+2} = F_n + F_{n+1}$, with $F_0 = 0$, $F_1 = 1$. The Fibonacci numbers can be also expressed by Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The Lucas numbers L_n satisfy the basic Fibonacci recurrence but $L_0 = 2$, $L_1 = 1$ and therefore $L_n = \alpha^n + \beta^n$.

* Tel.: +00420493 331 111; fax: +00420495 513 890.
E-mail address: Pavel.Trojovsky@uhk.cz.

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Thus, the Fibonacci coefficients can be expressed for integers $n \geqslant k \geqslant 1$ as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$$

with $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$ for $n < k$. Gould in [2] gives their recurrence formula in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = F_{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix} + F_{n-k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

A generalization of this recurrence for the general second-order linear recurrence we can find for example in [5]. As a special case of Horadam's result in [6] it is possible to get the following relation for the generating function $f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$ of the kth power of Fibonacci numbers

$$f_k(x) = \frac{\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{j(j+1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^k x^i}{\sum_{i=0}^{k+1} (-1)^{i(i+1)/2} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i}. \tag{1}$$

Shannon obtained in [9] some special forms of the generating function $f_k(x)$. We obtain for any odd integer k that

$$f_k(x) = 5^{-(k-1)/2} \sum_{j=0}^{(k-1)/2} \binom{k}{j} \frac{F_{k-2j} x}{1 - (-1)^j L_{k-2j} x - x^2} \tag{2}$$

and for any even integer k that

$$f_k(x) = 5^{-k/2} \sum_{j=0}^{(k-2)/2} (-1)^j \binom{k}{j} \frac{2 - (-1)^j L_{k-2j} x}{1 - (-1)^j L_{k-2j} x + x^2} + \binom{k}{\frac{k}{2}} \frac{(-5)^{-k/2}}{1 - (-1)^{k/2} x} \tag{3}$$

after simplification of one of Shannon's results.

The denominator of (1) is the polynomial $D_{k+1}(x) = \sum_{i=0}^{k+1} d_i x^i$, where coefficients $d_i = (-1)^{(i/2)(i+1)} \left[\begin{smallmatrix} k+1 \\ i \end{smallmatrix}\right]$ are terms of the sequence which was named as "signed Fibonomial triangle" in Sloane's on-line encyclopedia of integer sequences with ID Number A055870. Probably one of the most famous identities for the Fibonomial coefficients,

$$\sum_{j=0}^{k+1} (-1)^{(j/2)(j+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{n-j}^k = 0,$$

where n, k are any positive integers such that $n \geqslant k+1$, which corresponds to the sum in the numerator of the generating function (1) for $i \geqslant k+1$, is given in the encyclopedia (see [10]). From (1)–(3) we get the following generating functions of d_i:

$$D^{(\mathrm{o})}(x) = \prod_{j=0}^{(k-1)/2} (1 - (-1)^j L_{k-2j} x - x^2) \tag{4}$$

for any odd positive integer k and

$$D^{(\mathrm{e})}(x) = (1 - (-1)^{k/2} x) \prod_{j=0}^{(k/2)-1} (1 - (-1)^j L_{k-2j} x + x^2) \tag{5}$$

for any even positive integer k.