Information Processing Letters 116 (2016) 649–652

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Complexity of a disjoint matching problem on bipartite graphs

Gregory J. Puleo

Coordinated Science Lab, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA

ARTICLE INFO

Article history:
Received 26 June 2015
Received in revised form 18 April 2016
Accepted 31 May 2016
Available online 3 June 2016
Communicated by R. Uehara

Keywords:
Matchings
Edge coloring
Computational complexity
NP-hardness
Bipartite graphs

ABSTRACT

We consider the following question: given an (X, Y)-bigraph G and a set $S \subseteq X$, does G contain two disjoint matchings M_1 and M_2 such that M_1 saturates X and M_2 saturates S? When $|S| \geq |X| - 1$, this question is solvable by finding an appropriate factor of the graph. In contrast, we show that when S is allowed to be an arbitrary subset of X, the problem is NP-hard.

1. Introduction

A *matching* in a graph G is a set of pairwise disjoint edges. A matching *covers* a vertex $v \in V(G)$ if v lies in some edge of the matching, and a matching *saturates* a set $S \subseteq V(G)$ if it covers every vertex of S.

An (X, Y)-bigraph is a bipartite graph with partite sets X and Y. The fundamental result of matching theory is Hall's Theorem [5], which states that an (X, Y)-bigraph contains a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. While Hall's Theorem does not immediately suggest an efficient algorithm for finding a maximum matching, such algorithms have been discovered and are well-known [1,6].

A natural way to extend Hall's Theorem is to ask for necessary and sufficient conditions under which *multiple* disjoint matchings can be found. This approach was taken by Lebensold, who obtained the following generalization of Hall's Theorem.

Theorem 1.1 *(Lebensold [9]). An (X, Y)-bigraph has k disjoint matchings, each saturating X, if and only if*

$$\sum_{y \in Y} \min\{k, |N(y) \cap S|\} \geq k|S| \tag{1}$$

for all $S \subseteq X$.

When $k = 1$, the left side of (1) is just $|N(S)|$, so Theorem 1.1 contains Hall's Theorem as a special case. As observed by Brualdi, Theorem 1.1 is equivalent to a theorem of Fulkerson [3] about disjoint permutations of 0, 1-matrices. Theorem 1.1 is also a special case of Lovasz's (g, f)-factor theorem [10]. Like Hall's Theorem, Theorem 1.1 does not immediately suggest an efficient algorithm, but efficient algorithms exist for solving the (g, f)-factor problem [4], and these algorithms can be applied to find the desired k disjoint matchings. We discuss the algorithmic aspects further in Section 4.

A different extension was considered by Frieze [2], who considered the following problem:

Disjoint Matchings (DM)
Input: Two (X, Y)-bigraphs G_1, G_2 on the same vertex set.
Question: Are there matchings $M_1 \subseteq G_1$, $M_2 \subseteq G_2$ such that $M_1 \cap M_2 = \emptyset$ and each M_i saturates X?

When $G_1 = G_2$, this problem is just the $k = 2$ case of the problem considered by Lebensold, and is therefore polyno-

E-mail address: puleo@illinois.edu.

http://dx.doi.org/10.1016/j.ipl.2016.05.005
0020-0190/

mially solvable. On the other hand, Frieze proved that the Disjoint Matchings problem is NP-hard in general.

In this paper, we consider the following disjoint-matching problem, which can be naturally viewed as a restricted case of the Disjoint Matchings problem:

Single-Graph Disjoint Matchings (SDM)
Input: An (X, Y)-bigraph G and a vertex set $S \subseteq X$.
Question: Are there matchings $M_1, M_2 \subseteq G$ such that $M_1 \cap M_2 = \emptyset$, M_1 saturates X, and M_2 saturates S?

We call such a pair (M_1, M_2) an *S-pair*. When $S = X$, this problem is also equivalent to the $k = 2$ case of Lebensold's problem. The problem SDM is similar to a problem considered by Kamalian and Mkrtchyan [7], who proved that the following problem is NP-hard:

Residual Matching
Input: An (X, Y)-bigraph G and a nonnegative integer k.
Question: Are there matchings $M_1, M_2 \subseteq G$ such that $M_1 \cap M_2 = \emptyset$, M_1 is a maximum matching, and $|M_2| \geq k$?

When G has a perfect matching, we can think of the Residual Matching problem as asking whether there is *some* $S \subseteq X$ with $|S| = k$ such that G has an S-pair. In contrast, the SDM problem asks whether some *particular* S admits an S-pair. Since k is part of the input to the Residual Matching problem, it is *a priori* possible that SDM could be polynomially solvable while the Residual Matching problem is NP-hard, since one might need to check exponentially many candidate sets S.

In Section 2, we give a quick reduction from SDM to DM, justifying the view of SDM as a special case of DM, and in Section 3 we show that SDM is NP-hard, thereby strengthening Frieze's result. In Section 4 we show that SDM is polynomially solvable under the additional restriction $|S| \geq |X| - 1$.

2. Reducing SDM to DM

In this section, we show that any instance of SDM with $|S| < |X| - 1$ reduces naturally to an instance of DM. Since SDM-instances with $|S| \geq |X| - 1$ are polynomially solvable, as we show in Section 4, this justifies the claim that SDM is a special case of DM.

Theorem 2.1. *Let G be an (X, Y)-bigraph and let $S \subseteq V(G)$ with $|S| < |X| - 1$. Construct graphs G_1, G_2 as follows:*

$$V(G_1) = V(G_2) = V(G),$$
$$E(G_1) = E(G),$$
$$E(G_2) = E(G) \cup \{xy : x \in X - S, \ y \in Y\}.$$

The graph G has an S-pair if and only if there are disjoint matchings M_1, M_2 contained in G_1, G_2 respectively, each saturating X.

Proof. If $|Y| < |X|$, then it is clear that G has no S-pair and that G_1, G_2 do not have matchings that saturate X, so assume that $|Y| \geq |X|$.

First suppose that M_1, M_2 are disjoint matchings contained in G_1, G_2 respectively, each saturating X. Let $M_1' = M_1$ and let $M_2' = \{e \in M_2 : e \cap X \subseteq S\}$. It is clear that (M_1', M_2') is an S-pair.

Now suppose that we are given an S-pair (M_1', M_2'). In order to obtain the matchings M_1, M_2 in G_1, G_2 as needed, we need to enlarge M_2' so that it saturates all of X, rather than only saturating S. Let $Y' = \{y \in Y : y \notin V(M_2')\}$, and let $H = G_2[(X - S) \cup Y'] - M_1'$.

We claim that H has a matching that saturates $X - S$, and prove this by verifying Hall's Condition. Let any $X_0 \subseteq X - S$ be given. If $|X_0| = 1$, say $X_0 = \{x_0\}$, then $N_H(X_0)$ contains all of Y' except possibly the mate of x_0 in M_1. Hence

$$|N_H(X_0)| \geq |Y'| - 1 = |Y| - |S| - 1$$
$$\geq |X| - |S| - 1 \geq 1 = |X_0|,$$

as desired. On the other hand, if $|X_0| \geq 2$, then $N_H(X_0)$ contains all of Y', so that

$$|N_H(X_0)| = |Y'| = |Y| - |S| \geq |X| - |S| \geq |X_0|.$$

Hence Hall's Condition holds for H. Now let M be a perfect matching in H, let $M_1 = M_1'$, and let $M_2 = M_2' \cup M$. By construction, M_2 is a matching in G_2 that saturates X. It is clear that $M_1 \cap M_2 = \emptyset$, since the edges in M_1' were omitted from H. Hence M_1 and M_2 are as desired. □

3. Finding two matchings is NP-hard

Given an instance (G, S) of SDM, we call a pair of matchings (M_1, M_2) satisfying the desired condition an *S-pair*. When G' is a subgraph of G and $S' = S \cap V(G')$, we say that an S-pair (M_1, M_2) *contains* an S'-pair (M_1', M_2') if $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$.

We prove that SDM is NP-hard via a reduction from 3SAT. Let $c_1, \dots, c_s$ be the clauses and $\theta_1, \dots, \theta_t$ be the variables of an arbitrary 3SAT instance. We define a graph G as follows.

For each variable θ_i, let H_i be a copy of the cycle C_{4s}, with vertices $v_{i,1}, \dots, v_{i,4s}$ written in order. Define

$$X_i = \{v_{i,j} : j \text{ is even}\},$$
$$S_i = \{v_{i,j} : j \equiv 2 \pmod 4\}.$$

Since H_i is an even cycle, it has exactly two perfect matchings, one containing the edge $v_{i,1}v_{i,2}$ and the other containing the edge $v_{i,2}v_{i,3}$. In an S_i-pair (M_1, M_2) for H_i, we have $v_{i,1}v_{i,2} \in M_1$ if and only if $v_{i,2}v_{i,3} \in M_2$, and the same argument holds for the other vertices of S_i. Thus, H_i has only two possible S_i-pairs, illustrated in Fig. 1. We call these pairs the *true pair* and *false pair* for H_i.

In the full graph G, we will not add any new edges incident to the vertices of X_i, so it will still be the case that any S-pair in the full graph induces either the true pair or the false pair in H_i. We use these pairs to encode the truth values of the corresponding 3SAT-variables.

For each clause c_k, let L_k be a copy of K_2, with vertices w_k, z_k. Let $G = \left(\bigcup_j H_j\right) \cup \left(\bigcup_k L_k\right)$. Add edges to G as follows: if the variable θ_i appears positively in the clause c_k,