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Existence of home states in Petri nets is decidable

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ABSTRACT

We show that the problem whether a given Petri net has a home state (a marking reachable from every reachable marking) is decidable, and at least as hard as the reachability problem.

1. Introduction

Frequently, dynamic systems must have "home states", which are defined as states that can be reached from whichever state the system might be in. In various electronic devices, home states may be entered automatically after periods of inactivity, or may be forced to be reached by pushing a "reset" button. In self-stabilising systems [3], failure states can be recovered from automatically, preferably ending up in regular, non-erroneous home states. In Markov chain theory, home states are called "essential" states [2], a particularly important class being that of the "recurrent" states.

The main two decision problems concerning home states are (1) given a dynamic system S and a state q, is q a home state of S? and (2) given a system S, does it have a home state? We call them the home state problem (HSP) and the home state existence problem (HSEP), respectively.

For finite-state systems, both HSP and HSEP are trivially decidable, but this is no longer true for models which may have an infinite state space, like Petri nets. For Petri net models, HSP (and, in fact, a more general problem) was shown decidable in [5,6], but our knowledge about HSEP is more limited: the only result was obtained in [1], where it was shown that all live and bounded free-choice nets have home states, while live and bounded asymmetric-choice nets may not. HSEP is explicitly mentioned as an open problem in H. Wimmel's compilation of open problems in Petri net theory [12].

In the first part of the paper we show that HSEP is decidable, and provide an algorithm that constructs a home state whenever there is one. The algorithm combines the decision procedure for HSP described in [5] with a more recent result showing that the mutual reachability relation for Petri nets (the relation containing the pairs of markings of a net that are reachable from each other) is effectively semilinear [10]. In the second part of the paper we show that the reachability problem of Petri nets can be reduced polynomially to HSEP. This underlines the hardness of HSEP. In the concluding section of the paper,

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we mention some related problems whose decidability remains open.

2. Basic concepts

We assume familiarity with elementary notions of Petri nets [11], such as the notation $N = (P, T, F)$ for a net with places P, transitions T, and arcs F. The set of N's markings is $\mathbb{N}^P$, and its initial marking (if one exists) is usually denoted by M_0. A marking $M' \in \mathbb{N}^P$ is reachable from a marking $M \in \mathbb{N}^P$ by a firing sequence $\tau \in T^*$, also denoted by $M \xrightarrow{\tau} M'$, if τ leads from M to M'. The set of all markings reachable from M is denoted by $[M\rangle$. We assume Petri nets to be finite. Observe that an initially marked finite net (N, M_0) can be unbounded, thus generating an infinite state space (i.e., an infinite set $[M_0\rangle$).

Definition 1. Let (N, M_0) be an initially marked net. A set of markings $\mathcal{M}$ of N is a *home space* of (N, M_0) if for every marking M which is reachable from M_0, some marking in $\mathcal{M}$ is reachable from M. A marking M is called a *home state* of (N, M_0) if $\{M\}$ is a home space of (N, M_0).

Observe that a set of markings can be a home space of (N, M_0), without necessarily containing a home state of (N, M_0). Also observe that $\emptyset$ is never a home space while $[M_0\rangle$ always is.

Definition 2. Let N be a net. Two markings M, M' of N are *mutually reachable* if M' is reachable from M and vice versa. The *mutual reachability relation* of N is the set containing the pairs (M, M') of markings of N such that M and M' are mutually reachable.

Note that this defines an equivalence on $\mathbb{N}^P$ which does not depend on any initial marking, but only on the structure of N.

Definition 3. Let N be a net. A marking M of N is a *bottom marking of* N if for every marking M' reachable from M, the markings M and M' are mutually reachable.

Note that a dead marking (enabling no transition) is automatically a bottom marking. Also, observe that bottom markings of N are related to home states of a marked net (N, M_0), with the same underlying net N. If M is a home state of a marked Petri net (N, M_0), then M is reachable from M_0, and it is a bottom marking of N. In that case, any other bottom marking reachable from $[M_0\rangle$ is also a home state. However, a marking M can be reachable from M_0 and also be a bottom marking of N, without there necessarily being a home state of (N, M_0).

The bottom markings of N are computationally more amenable than the home states of (N, M_0) because (as it will turn out) they are semilinear in the following sense.

Definition 4. Let $k \in \mathbb{N}$. A set $\mathcal{M} \subseteq \mathbb{N}^k$ is *linear* if there exists a root vector $\rho \in \mathbb{N}^k$ and a finite set of periods $\Pi = \{\pi_1, \ldots, \pi_n\} \subseteq \mathbb{N}^k$ such that

$$\mathcal{M} = \bigcup_{\lambda_1,\ldots,\lambda_n \in \mathbb{N}} \{M \in \mathbb{N}^k \mid M = \rho + \sum_{i=1}^{n} \lambda_i \pi_i\}$$

and *semilinear* if $\mathcal{M} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_m$ for m linear sets $\mathcal{M}_1, \ldots, \mathcal{M}_m$.

We denote by $(\rho; \Pi)$ the linear set with root vector ρ and period set Π.

A subset of $\mathbb{N}^k$ (for some $k \in \mathbb{N}$) is semilinear if and only if it is Presburger definable [8]. Semilinearity and Presburger definability extend to subsets of $\mathbb{N}^k \times \mathbb{N}^k$ using $\mathbb{N}^k \times \mathbb{N}^k = \mathbb{N}^{2k}$.

A set $\mathcal{M}$ of markings of a net N with k places is *effectively semilinear* if there is an algorithm that on input N returns root vectors $\rho_1, \ldots, \rho_m \in \mathbb{N}^k$ and period sets $\Pi_1, \ldots, \Pi_m \subseteq \mathbb{N}^k$ such that $\mathcal{M} = \bigcup_{i=1}^{m} (\rho_i; \Pi_i)$. Similarly, $\mathcal{M}$ is *effectively definable in Presburger arithmetic* if there is an algorithm that on input N returns a formula of Presburger arithmetic defining $\mathcal{M}$. Effectively semilinear and Presburger definable relations on the markings of N are defined analogously, by identifying $\mathbb{N}^k \times \mathbb{N}^k$ with $\mathbb{N}^{2k}$. By [8], effective semilinearity and effective definability in Presburger arithmetic coincide.

The home state existence problem is defined as follows:

$$\text{HSEP:} \begin{cases} \textbf{Given:} & \text{An initially marked Petri net } (N, M_0). \\ \textbf{Decide:} & \text{Is there a home state } M \text{ of } (N, M_0)? \end{cases}$$

3. Decidability of HSEP

To commence the proof of the decidability of HSEP, we recall the following strong result by J. Leroux:

Theorem 1. *(See [10].) For every Petri net N, the mutual reachability relation is effectively definable in Presburger arithmetic.*[2]

So, by [8], the mutual reachability relation is semilinear. This easily leads to the following result, already described in [4]. Since the proof is short, we give a sketch.

Theorem 2. *(See [4].) Let N be a net. The set of bottom markings of N is effectively semilinear.*

Proof. We show that the predicate $\mathtt{B}(M)$ associated to the set of bottom markings is effectively definable in Presburger arithmetic, and so semilinear. By Theorem 1, we can compute a Presburger predicate $\mathtt{MR}(M, M')$ associated to the mutual reachability relation. Now, we observe – using induction on the length of a firing sequence – that M is a bottom marking iff for every marking M' such that M and M' are mutually reachable, and for every M'' such that there is some $t \in T$ with $M' \xrightarrow{t} M''$, the markings M and M'' are also mutually reachable. Hence

[2] Observe the emphasis on *effective* definability. Definability in Presburger arithmetic follows from a result by Eilenberg and Schützenberger on commutative monoids [7].