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Linear-vertex kernel for the problem of packing r-stars into a graph without long induced paths

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ABSTRACT

Let integers $r \geq 2$ and $d \geq 3$ be fixed. Let $\mathcal{G}_d$ be the set of graphs with no induced path on d vertices. We study the problem of packing k vertex-disjoint copies of $K_{1,r}$ ($k \geq 2$) into a graph G from parameterized preprocessing, i.e., kernelization, point of view. We show that every graph $G \in \mathcal{G}_d$ can be reduced, in polynomial time, to a graph $G' \in \mathcal{G}_d$ with $O(k)$ vertices such that G has at least k vertex-disjoint copies of $K_{1,r}$ if and only if G' has. Such a result is known for arbitrary graphs G when $r = 2$ and we conjecture that it holds for every $r \geq 2$.

1. Introduction

For a fixed graph H, the problem of deciding whether a graph G has k vertex-disjoint copies of H is called H-Packing. The problem has many applications (see, e.g., [2,3,9]), but unfortunately it is almost always intractable. Indeed, Kirkpatrick and Hell [9] proved that if H contains a component with at least three vertices then H-Packing is NP-complete. Thus, approximation, parameterized, and exponential algorithms have been studied for H-Packing when H is a fixed graph, see, e.g., [2,6,7,12,13].

In this note, we will consider H-Packing when $H = K_{1,r}$ and study $K_{1,r}$-Packing from a parameterized preprocessing, i.e., kernelization, point of view.[1] Here k is the parameter. As a parameterized problem, $K_{1,r}$-Packing was first considered by Prieto and Sloper [12] who obtained an $O(k^2)$-vertex kernel for each $r \geq 2$ and a kernel with at most $15k$ vertices for $r = 2$. (Since the case $r = 1$ is polynomial-time solvable, we may restrict ourselves to $r \geq 2$.) The same result for $r = 2$ was proved by Fellows et al. [6] and it was improved to $7k$ by Wang et al. [13].

Fellows et al. [6] note that, using their approach, the bound of [12] on the number of vertices in a kernel for any $r \geq 3$ can likely be improved to subquadratic. We believe that, in fact, there is a linear-vertex kernel for every $r \geq 3$ and we prove Theorem 1 to support our conjecture. A path P in a graph G, is called *induced* if it is an induced subgraph of G. For an integer $d \geq 3$, let $\mathcal{G}_d$ denote the set of all graphs with no induced path on d vertices.

Theorem 1. *Let integers $r \geq 2$ and $d \geq 3$ be fixed. Then $K_{1,r}$-Packing restricted to graphs in $\mathcal{G}_d$, has a kernel with $O(k)$ vertices.*

Since d can be an arbitrary integer larger than two, Theorem 1 is on an ever increasing class of graphs which,

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[1] We provide basic definitions on parameterized algorithms and kernelization in the next section, for recent monographs, see [4,5]; [10,11] are recent survey papers on kernelization.

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in the “limit”, coincides with all graphs. To show that Theorem 1 is an optimal[2] result, in a sense, we prove that $K_{1,r}$-PACKING restricted to graphs in $\mathcal{G}_d$ is $\mathcal{NP}$-hard already for $d = 5$ and every fixed $r \geq 3$:

Theorem 2. *Let $r \geq 3$. It is $\mathcal{NP}$-hard to decide if the vertex set of a graph in $\mathcal{G}_5$ can be partitioned into vertex-disjoint copies of $K_{1,r}$.*

We cannot replace $\mathcal{G}_5$ by $\mathcal{G}_4$ (unless $\mathcal{NP} = \mathcal{P}$) due to the following assertion, whose proof is provided in the Appendix of [1].

Theorem 3. *Let $r \geq 3$ and $G \in \mathcal{G}_4$. We can find the maximal number of vertex-disjoint copies of $K_{1,r}$ in G in polynomial time.*

2. Terminology and notation

For a graph G, $V(G)$ ($E(G)$, respectively) denotes the vertex set (edge set, respectively) of G, $\Delta(G)$ denotes the maximum degree of G and n its number of vertices. For a vertex u and a vertex set X in G, $N(u) = \{v : uv \in E(G)\}$, $N[u] = N(u) \cup \{u\}$, $d(u) = |N(u)|$, $N_X(u) = N(u) \cap X$, $d_X(u) = |N_X(u)|$ and $G[X]$ is the subgraph of G induced by X. We call $K_{1,r}$ an *r-star*. We say a star *intersects* a vertex set if the star uses a vertex in the set. We use (G, k, r) to denote an instance of the r-star packing problem. If there are k vertex-disjoint r-stars in G, we say (G, k, r) is a YES-instance, and we write $G \in \star(k, r)$. Given disjoint vertex sets S, T and integers s, r, we say that S has *s r-stars in T* if there are s vertex-disjoint r-stars with centers in S and leaves in T.

A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ. A parameterized problem L is *fixed-parameter tractable* if the membership of an instance (I, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|I|^{O(1)}$ where f is a computable function of the *parameter* k only. Given a parameterized problem L, a *kernelization of L* is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') (the *kernel*) such that $(x, k) \in L$ if and only if $(x', k') \in L$ and $k' + |x'| \leq g(k)$ for some function g. It is well-known that a decidable parameterized problem L is fixed-parameter tractable if and only if it has a kernel. Kernels of small size are of main interest, due to applications.

3. Proof of Theorem 1

Note that the 1-star packing problem is the classic maximum matching problem and if $k = 1$, the r-star packing problem is equivalent to deciding whether $\Delta(G) \geq r$. Both of these problems can be solved in polynomial time. Henceforth, we assume $r, k > 1$.

A vertex u is called a *small vertex* if $\max\{d(v) : v \in N[u]\} < r$. A graph without a small vertex is a *simplified graph*.

We now give two reduction rules for an instance (G, k, r) of $K_{1,r}$-PACKING.

[2] If $K_{1,r}$-PACKING was polynomial time solvable, then it would have a kernel with $O(1)$ vertices.

Reduction Rule 1. *If graph G contains a small vertex v, then return the instance $(G - v, k, r)$.*

It is easy to observe that Reduction Rule 1 can be applied in polynomial time.

Let $G = (V, E)$ be a graph and let C, L be two vertex-disjoint subsets of V. The pair (C, L) is called a *constellation* if $G[C \cup L] \in \star(|C|, r)$ and there is no star $K_{1,r}$ intersecting L in the graph $G[V \setminus C]$.

Reduction Rule 2. *If (C, L) is a constellation, return the instance $(G[V \setminus (C \cup L)], k - |C|)$.*

It is easy to observe that Reduction Rule 2 can be applied in polynomial time, provided we are given a suitable constellation.

Lemma 1. *Reduction Rules 1 and 2 are safe.*

Proof. Clearly, a small vertex v cannot appear in any r-star. Therefore Reduction Rule 1 is safe as G and $G - v$ will contain the same number of r-stars.

To see that Reduction Rule 2 is safe, it is sufficient to show that $G \in \star(k, r)$ if and only if $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$. On the one hand, if $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$, the hypothesis $G[C \cup L] \in \star(|C|, r)$ implies $G \in \star(k, r)$. On the other hand, there are at most $|C|$ vertex-disjoint stars intersecting C. But by hypothesis, every star intersecting L also intersects C. We deduce that there are at most $|C|$ stars intersecting $C \cup L$, and so if $G \in \star(k, r)$, there are at least $k - |C|$ stars in $G[V - (C \cup L)]$: $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$. □

Note that as both rules modify a graph by deleting vertices, any graph G' that is derived from a graph $G \in \mathcal{G}_d$ by an application of Rules 1 or 2 is also in $\mathcal{G}_d$.

Recall the Expansion Lemma, which is a generalization of the well-known Hall's theorem.

Lemma 2 *(Expansion Lemma). [8] Let r be a positive integer, and let m be the size of the maximum matching in a bipartite graph G with vertex bipartition $X \cup Y$. If $|Y| > rm$, and there are no isolated vertices in Y, then there exist nonempty vertex sets $S \subseteq X$, $T \subseteq Y$ such that S has $|S|$ r-stars in T and no vertex in T has a neighbor outside S. Furthermore, the sets S, T can be found in polynomial time in the size of G.*

Henceforth, we will use the following modified version of the expansion lemma.

Lemma 3 *(Modified Expansion Lemma). Let r be a positive integer, and let m be the size of the maximum matching in a bipartite graph G with vertex bipartition $X \cup Y$. If $|Y| > rm$, and there are no isolated vertices in Y, then there exists a polynomial algorithm(in the size of G) which returns a partition $X = A_1 \cup B_1$, $Y = A_2 \cup B_2$, such that B_1 has $|B_1|$ r-stars in B_2, $E(A_1, B_2) = \emptyset$, and $|A_2| \leq r|A_1|$.*

Proof. Apply the Expansion Lemma on graph G to get nonempty vertex sets $S \subseteq X, T \subseteq Y$ such that S has $|S|$