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# On the complexity of exchanging ☆

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ABSTRACT

We analyze the computational complexity of the problem of deciding whether, for a given simple game, there exists the possibility of rearranging the participants in a set of $j$ given losing coalitions into a set of $j$ winning coalitions. We also look at the problem of turning winning coalitions into losing coalitions. We analyze the problem when the simple game is represented by a list of wining, losing, minimal winning or maximal loosing coalitions.



## 1. Introduction

Simple games cover voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. In these systems, each voter responds with a vote of yea and nay. Democratic societies and international organizations use a wide variety of complex rules to reach decisions. Examples, where it is not always easy to understand the consequences of the way voting is done, include the Electoral College to elect the President of the United States, the United Nations Security Council, the governance structure of the World Bank, the International Monetary Fund, the European Union Council of Ministers, the national governments of many countries, the councils in several counties, and the system to elect the major in cities or villages of many countries. Another source of examples comes from economic enterprises whose owners are shareholders of the society and divide profits or losses proportionally to the numbers of stocks they posses, but make decisions by voting according to a pre-defined rule (i.e., an absolute majority rule or a qualified majority rule). See [11,12] for a thorough presentation of theses and other examples. Such systems have been analyzed as *simple games*.

**Definition 1.** A *simple game* $\Gamma$ is a pair $(N, \mathcal{W})$ in which $N = \{1, 2, \ldots, n\}$ and $\mathcal{W}$ is a collection of subsets of $N$ that satisfies: *(1)* $N \in \mathcal{W}$, *(2)* $\emptyset \notin \mathcal{W}$ and *(3)* the *monotonicity* property: $S \in \mathcal{W}$ and $S \subseteq T \subseteq N$ implies $T \in \mathcal{W}$.

The subsets of $N$ are called *coalitions*, the coalitions in $\mathcal{W}$ are called *winning coalitions*, and the coalitions that are not winning are called *losing coalitions* (noted by $\mathcal{L}$). Moreover, we say that a coalition is minimal winning (maximal losing) if it is a winning (losing) coalition all of whose proper subsets (supersets) are losing (winning). Because of monotonicity, any simple game is completely determined by its set of minimal winning (maximal losing) coalitions

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denoted by $\mathcal{W}^m$ ($\mathcal{L}^M$). Note that a description of a simple game $\Gamma$ can be given by $(N, \mathcal{X})$, where $\mathcal{X}$ is $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ or $\mathcal{L}^M$, see [12]. We focus on the process of exchanging or trading where a motivating example is the following:

**Example 1.** Consider two English football clubs that are in trouble and in danger of leaving Premier League. Maybe the two clubs could trade with each other and exchange players so they both could avoid relegation. We consider the complexity of figuring out if such an exchange is possible for various ways of knowing what it takes to form a strong team that is able to stay in Premier League. This can be viewed as a simple game where a winning coalition corresponds to a strong team of players.

The considered property is the so called $j$-trade property for simple games. Loosely speaking, a simple game is $j$-trade if it is possible to rearrange the players in a set of $j$ winning (losing) coalitions into a set of $j$ losing (winning) coalitions, in such a way that the total number of occurrences of each player is the same in both sets. Thus, it is possible to go from one set to the other via participant trades. This notion was introduced by Taylor and Zwicker [12] in order to obtain a characterization of the weighted games, a subfamily of simple games. Recall that any simple game can be expressed as the intersection of weighted simple games. This leads to the definition of the *dimension* concept, the minimum number of required weighted games whose intersection represents the simple game [2,6,3]. Due to this fact, the problem of deciding whether a simple game is weighted has been of interest in several contexts. With respect to tradeness, it is known that a simple game is weighted if and only if it is not $j$-trade for any non-negative integer $j$ [12]. Freixas et al. [5] studied the computational complexity of deciding whether a simple game is weighted among other decision problems for simple games. In particular, they showed that deciding whether a simple game is weighted is polynomial time solvable when the game is given by an explicit listing of one of the families $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$, $\mathcal{L}^M$. On the other hand, the $j$-trade concept was also redefined as $j$-invariant-trade of simple games [4] and extended as $(j,k)$-simple games [7].

Here we provide a definition of $j$-trade that uses a formalism that differ from the classic one for $j$-trade *robustness* applied to a simple game (see [1,12,4]) in order to ease the proofs of our new results.

**Definition 2.** Given a simple game $\Gamma$, a *$j$-trade application* is a set of $2j$ coalitions $(S_1, \ldots, S_{2j})$ such that $\exists I \subseteq \{1, \ldots, 2j\}$ that satisfies:

1. $|I| = j$
2. $\forall i \in \{1, \ldots, 2j\}$, $S_i \in \mathcal{W} \iff i \in I$
3. $\forall p \in N$, $|\{i \in I : p \in S_i\}| = |\{i \in \{1, \ldots, 2j\} \setminus I : p \in S_i\}|$

**Definition 3.** A simple game $\Gamma$ is *$j$-trade* if it admits a $j$-trade application.

**Example 2.** The simple game defined by $(N, \mathcal{W}^m) = (\{1,2,3,4\}, \{\{1,3\},\{2,4\}\})$ is 2-trade because it admits a 2-trade application. For instance, we can consider the following set of coalitions $(\{1,3\},\{2,4\},\{1,2\},\{3,4\})$ where $\{1,3\},\{2,4\} \in \mathcal{W}$, but $\{1,2\},\{3,4\} \in \mathcal{L}$.

**Example 3.** It is easy to generate a simple game that will be $2j$-trade, for an integer $j$. For instance, we can take the simple game $(N, \mathcal{W}^m)$ where $N = \{1, \ldots, 2j\}$ and $\mathcal{W}^m = \{\{i, i+1\} \mid i \in 1, 3, 5, \ldots, 2j-1\}$. It is clear that coalitions $L_i = \{i, i+1\}$, for all $i \in \{2, 4, 6, \ldots, 2j-2\}$, and $L_{2j} = \{1, 2j\}$ are losing. Thus, the set of $2j$ coalitions $\mathcal{W}^m \cup \left(\cup_{i=1}^{j} L_{2i}\right)$ generates a $j$-trade application.

**Definition 4.** A simple game $\Gamma$ is *$j$-trade robust* if it is not *$j$-trade*.

Before formally defining the *decision problems* we focus on, we consider two functions $\alpha$ and $\beta$ associating games with various types of sets of coalitions. The allowed types are the following $\alpha(\Gamma) \in \{\mathcal{W}, \mathcal{L}, \mathcal{W}^m, \mathcal{L}^M\}$ and $\beta(\Gamma) \in \{\mathcal{W}, \mathcal{L}\}$, respectively. Moreover, given the $\beta$ application we consider the function $\overline{\beta}$ that provides the *complementary* type with respect to the function $\beta$.

$$\overline{\beta}(\Gamma) = \begin{cases} \mathcal{W}, & \text{if } \beta(\Gamma) = \mathcal{L} \\ \mathcal{L}, & \text{if } \beta(\Gamma) = \mathcal{W} \end{cases}$$

Now we can state the definition of the considered computational problems, observe that the value of $\alpha$ provides the type of coalitions used in the representation of the input game while the $\beta$ function indicates the type of the coalitions to be exchanged.

**Definition 5.** The *$(\alpha, \beta, j)$-trade* problem, where $j \in \mathbb{N}$, is

> `Input:` A simple game $\Gamma$ given by $(N, \alpha(\Gamma))$ and $j$ coalitions $S_1, \ldots, S_j \in \beta(\Gamma)$.
> `Question:` Do there exist $S_{j+1}, \ldots, S_{2j} \in \overline{\beta}(\Gamma)$ such that $(S_1, \ldots, S_{2j})$ is a $j$-trade application?

**Definition 6.** The $(\alpha, \beta)$-`Trade` problem is the $(\alpha, \beta, 2)$-trade problem.

In the remaining part of the paper we analyze the computational complexity of the above problems. Table 1 summarizes all results about the $(\alpha, \beta)$-`Trade` problem. We present first the results for the $(\alpha, \beta)$-`Trade` problem and then the results for the general case. We finalize with some conclusions and open problems.

## 2. The computational complexity of trading two given coalitions

We present first the types for which the $(\alpha, \beta)$-`Trade` problems are polynomial time solvable.

**Proposition 1.** *The $(\alpha, \beta)$-`Trade` problem is polynomially time solvable when $\alpha(\Gamma) \in \{\mathcal{W}, \mathcal{W}^m, \mathcal{L}\}$ and $\beta(\Gamma) = \mathcal{L}$.*

**Proof.** We analyze each case separately. Let $S_1, S_2$ be two coalitions and assume that both are of type $\beta(\Gamma) = \mathcal{L}$.