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Towards optimal kernel for edge-disjoint triangle packing

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ABSTRACT

We study the kernelization of the EDGE-DISJOINT TRIANGLE PACKING (ETP) problem, in which we are asked to find k edge-disjoint triangles in an undirected graph. ETP is known to be fixed-parameter tractable with a $4k$ vertex kernel. The kernelization first finds a maximal triangle packing which contains at most $3k$ vertices, then the reduction rules are used to bound the size of the remaining graph. The constant in the kernel size is so small that a natural question arises: Could $4k$ be already the optimal vertex kernel size for this problem? In this paper, we answer the question negatively by deriving an improved $3.5k$ vertex kernel for the problem.

1. Introduction

Problem kernelization has been recognized as one of the most significant contributions of parameterized complexity to practical computing. Intuitively, a kernelization shrinks an instance of a parameterized problem in polynomial time to an equivalent instance whose size is bounded by a function of the parameter. Recall that a parameterized problem contains a main part I and a parameter k. In a formal way, a *kernelization* of a parameterized problem Q is a polynomial-time algorithm that takes as input an instance (I, k) of Q and outputs a new instance (I', k') of Q such that (1) the size of the new instance (I', k') is bounded by a function f of k; (2) $k' \leqslant k$; and (3) (I, k) is a yes-instance if and only if (I', k') is a yes-instance. The new instance is called the *kernel* of the problem, and the function f is the size of the kernel. In the context of graph problems, many research papers use the term vertex kernel size to refer to the kernel size counted as the number of vertices in the kernel, see, e.g., [8,9,12]. In this paper, we adopt this term. In particular, if f is linear, we have a *linear kernel*. If the kernel size is small, the problem may be solvable in an acceptable time even if it is NP-hard. Based on this observation kernelizations, especially linear kernelizations, are used in various ways, for instance, as a preprocessing procedure, combined with heuristic or approximation algorithms, to solve the real-world problems [4,11]. This motivates researchers to explore an improved kernel, in particular an improved linear kernel, once a kernel for a problem is derived. For example, for the PLANAR CONNECTED VERTEX COVER problem, which started with a vertex kernel of size at most $14k$ [9] and then was improved to $4k$ [17], a vertex kernel of size at most $11/3k$ has been shown very recently [12]. Other examples include CLUSTER EDITING [2], MAXIMUM INTERNAL SPANNING TREE [8], PLANAR DOMINATING SET [5], etc.

Kernelization also plays a significant role in the theoretical study of parameterized complexity. It is well known that a problem is fixed-parameter tractable (FPT) if and only if it admits a kernelization (see Theorem 1.39 in [7] or Proposition 7.2 in [16] for formal proofs). Recall that a parameterized problem is FPT if it is solvable in $O(f(k) \cdot |I|^{O(1)})$ time, where (I, k) is any instance of the problem. Moreover, the recently derived kernel lower bound techniques have built connections between kernel lower bounds and several well-established complexity conjectures. In a concrete way, researchers have shown that the

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existence of special kernels of certain problems conflicts with several complexity conjectures. For instance, for the Vertex Cover problem, a kernel with $O(k^{2-\epsilon})$ edges for any $\varepsilon > 0$ would imply coNP $\subseteq$ NP/poly [6]. We refer to [1,13,15] for more information on this topic.

In this paper, we consider the kernelization of the Edge-Disjoint Triangle Packing problem. The formal definition of the problem is as follows.

Edge-Disjoint Triangle Packing (Etp)
Input: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Are there at least k edge-disjoint triangles in G?

Etp is known to be NP-hard [10], even when restricted to planar graphs with maximum degree 5 [3]. In [14], the authors gave a $4k$ vertex kernel for the problem. The kernelization first carries out several reduction rules to remove some specific structures. Then, a maximal triangle packing $\mathcal{P}$ is found in polynomial time (a triangle packing is a set of pairwise edge-disjoint triangles). If $\mathcal{P}$ contains k triangles, the problem is solved. Otherwise, $V(\mathcal{P})$ contains at most $3k$ vertices. Finally, the number of vertices not in $V(\mathcal{P})$ is bounded by k according to the reduction rules. In this paper, we propose a simple kernelization for Etp which results in a $3.5k$ vertex kernel, improving the previous $4k$ bound. To this end, we introduce new reduction rules, so that after removing several structures based on these reduction rules, we can always find a maximal triangle packing which involves at most $2.5k$ vertices in polynomial time. More concretely, we can find a maximal triangle packing $\mathcal{P}$ in which every triangle shares at least one vertex with another triangle in $\mathcal{P}$. The terminologies used in this paper are defined as follows.

We consider only undirected simple graphs. For a graph $G = (V, E)$, let $V(G)$ denote the vertex set of G and $E(G)$ the edge set of G. For $V' \subseteq V$, the *subgraph induced by* V', denoted $G[V']$, is the subgraph of G with vertex set V' and edge set $\{(v, u) \mid v, u \in V', (v, u) \in E\}$. The graph obtained from G by removing V' (resp. $E' \subseteq E$) is denoted by $G \setminus V'$ (resp. $G \setminus E'$). For a vertex v, $N(v)$ is the open neighborhood of v, i.e., $N(v) = \{u \in V \mid (u, v) \in E\}$. An *independent set* is a subset of pairwise non-adjacent vertices. A *complete graph* of size n, denoted K_n, is a graph with n vertices in which every two vertices are adjacent. The complete graph K_3 is also called a *triangle*. We use (v, u, w) to denote the triangle formed by the vertices v, u and w. For a vertex v and an edge (u, w), we say v *spans* (u, w) if v forms a triangle with u and w. A *triangle packing* is a set of pairwise edge-disjoint triangles. For a triangle packing $\mathcal{P}$, we use $V(\mathcal{P})$ to denote the set of vertices contained in some triangle in $\mathcal{P}$. For a subset $V' \subseteq V$, we use $S(V')$ to denote the set of edges that are not in $G[V']$ but are spanned by some vertex in V', that is, $S(V') = \{(u, w) \mid u, w \notin V' \text{ and } \exists v \in V' \text{ s.t. } v \text{ spans } (u, w)\}$. For a subset $E' \subseteq E$, we use $\bar{S}(E')$ to denote the set of vertices that are not in $V(E')$ but span some edge in E', that is, $\bar{S}(E') = \{v \mid v \notin V(E') \text{ and } \exists e \in E' \text{ s.t. } v \text{ spans } e\}$. Here, $V(E')$ is the set of endpoints of the edges in E'.

2. Reduction rules

In the following, we introduce the reduction rules. We assume that before the jth rule is introduced, there is no ith rule can be applied to the graph for all $i < j$. The correctness of these reduction rules are explained in Lemma 3.

Rule 1. *Remove all the vertices and edges that are not included in any triangle.*

Rule 2. *If there is a triangle formed by v, u and w such that $N(v) \cap N(u) = \{w\}$ and $N(v) \cap N(w) = \{u\}$, then remove $(v, u), (u, w), (v, w)$ and decrease the parameter by* 1.

Rule 3. *If there is a K_4 formed by v, x, y and z such that $N(x) \cap N(y) = \{z, v\}$, $N(x) \cap N(v) = \{z, y\}$ and $N(v) \cap N(y) = \{z, x\}$, then remove $(v, x), (x, y), (v, y)$ and decrease the parameter by* 1.

The fat-head crown decomposition was studied in [14]. A *fat-head crown decomposition* of a graph $G = (V, E)$ is a triple (C, H, X) such that C and X form a partition of $V(G)$, $H \subseteq E(G)$ and the following properties hold:

1. C is an independent set
2. H is the set of edges spanned by C, i.e., $H = S(C)$
3. $V(H)$, the endpoints of the edges in H, forms a separator such that there are no edges from C to $X \setminus V(H)$.
4. there is a function f mapping each edge $(h_1, h_2) \in H$ to a unique vertex $u \in C$ which spans (h_1, h_2).

Rule 4. *If there is a fat-head crown decomposition (C, H, X) of G, then remove H and C from the graph and decrease the parameter by $|H|$.*

To use the reduction rule 4, we need a method to find out a fat-head crown decomposition if one exists. The following lemma, which has been studied in [14], provides such a method.

Lemma 1. *For a graph $G = (V, E)$ where each vertex and edge is included in at least one triangle, if there is an independent set $I \subseteq V$ such that $|I| \geqslant |S(I)|$, then there is a fat-head crown decomposition of G which can be found in polynomial time if given I.*

The correctness of the reduction rule 1 is easy to see. The correctness of the reduction rule 4 has been studied in [14]. In the following, we prove the correctness of the reduction rules 2 and 3. We say a triangle T_1 *blocks* another triangle T_2 if T_2 shares an edge with T_1. For a triangle T let $B(T)$ be the set of triangles that are blocked by T, that is, $B(T) = \{T' \mid T' \text{ is a triangle and } E(T) \cap E(T') \neq \emptyset\}$, and let $B[T] = B(T) \cup \{T\}$. The following lemma is useful.

Lemma 2. *For a triangle T, if all the triangles in $B[T]$ block one another, then there must be a maximum triangle packing containing T.*