Information Processing Letters 116 (2016) 299–303

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On the construction of all shortest node-disjoint paths in star networks

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ARTICLE INFO

Article history:
Received 4 April 2015
Received in revised form 22 October 2015
Accepted 4 November 2015
Available online 10 November 2015
Communicated by Tsan-sheng Hsu

Keywords:
Star network
Node-disjoint paths
Maximum matching
Optimization problem
Interconnection networks

ABSTRACT

Node-disjoint paths have played an important role in the study of routing, reliability, and fault tolerance of an interconnection network. In this paper, we give a sufficient condition, which can be verified in $O(mn^{1.5})$ time, for the existence of m node-disjoint shortest paths from one source node to other m (not necessarily distinct) target nodes, respectively, in an n-dimensional star network, where $m \leq n-1$ and $n \geq 3$. In addition, when the condition holds, the m node-disjoint shortest paths can be constructed in optimal $O(mn)$ time. By the aid of the sufficient condition, the probability of existence of all shortest node-disjoint paths was also evaluated for some special cases.

1. Introduction

Due to the advancement of hardware technology, it is now feasible to build a large-scale multiprocessor system consisting of hundreds or even thousands of processors. One crucial step in designing a multiprocessor system is to determine the topology of its interconnection network (network for short) in which nodes and links correspond to processors and communication channels, respectively. Among the network topologies proposed in the literature, the star network has received much attention from outstanding researchers [1,2,4,5,8,12,13,15].

An n-dimensional star network (n-star for short) consists of $n!$ nodes that are labeled with $n!$ n-digit permutations of $\{1, 2, \ldots, n\}$, and there is a link between two nodes if one of their labels can be obtained by swapping the first digit for another digit in the other's label. The degree and connectivity of an n-star are both $n-1$, where the *connectivity* of a network is the minimum number of nodes whose removal can make the network disconnected or trivial. Fig. 1 shows the structure of a 4-star. As recognized as alternatives to hypercube networks, star networks have enjoyed the popularity due to many of their attractive topological properties, including regularity, node symmetric, link symmetric, sublogarithmic degree and diameter, recursive construction, and strong resilience [1,4].

Node-disjoint paths have made themselves play an important role in the study of routing, reliability, and fault tolerance of a network because they can be used to avoid congestion, accelerate transmission rate, and provide alternative transmission routes. Two paths are *internally node-disjoint* (disjoint for short) if they do not share any common node except their end nodes. In order to reduce the transmission latency and cost, the maximal length and total length are required to be minimized, respectively, where the *length* of a path is the number of links in it. There are three kinds of disjoint paths, i.e., one-to-one, one-to-many, and many-to-many. Among them, one-to-many disjoint paths [3,7,14,16] from one node to other mutually distinct nodes were first studied in [14] where

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http://dx.doi.org/10.1016/j.ipl.2015.11.003

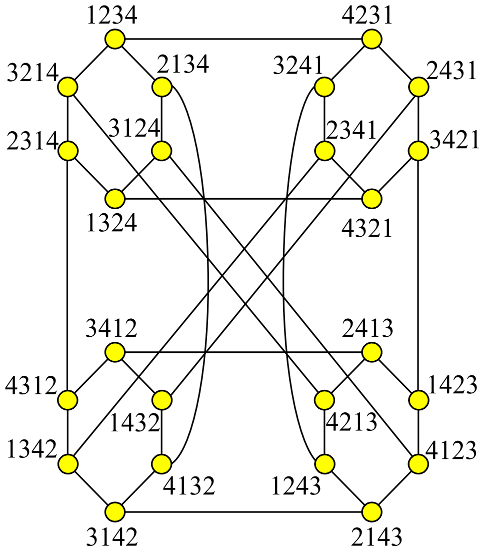

Fig. 1. The structure of a 4-star.

the Information Dispersal Algorithm (IDA for short) was proposed on the hypercube. By taking advantages of disjoint paths, the IDA has numerous potential applications to secure and fault-tolerant storage and transmission of information.

Recently, all shortest disjoint paths from one source node/vertex to other (not necessarily distinct) target nodes/vertices have been constructed in hypercubes [11], tori [9], and the Cayley graphs of abelian groups [10]. In this paper, we study the problem of constructing m disjoint shortest paths from one source node to other m (not necessarily distinct) target nodes in an n-star, where $m \le n-1$ and $n \ge 3$. A sufficient condition, which can be verified in $O(mn^{1.5})$ time, for the existence of the m disjoint shortest paths will be proposed. When the condition holds, the m disjoint shortest paths can be constructed in optimal $O(mn)$ time. In order to figure out the probability that there exist disjoint shortest paths in a star network, brute-force computations were carried out for verifying the sufficient condition on all the combinations of the source node and target nodes by running computer programs. In the situation that all of the source node and target nodes are mutually distinct, computational results have shown that the probability of the existence of the m disjoint shortest paths in the n-star is greater than 80%, 88%, 80%, 77%, and 92% for $(m,n) = (2,3), (2,4), (3,5), (4,6)$, and $(3,7)$, respectively.

The rest of this paper is organized as follows: In the next section, the construction of all shortest paths in an n-star is first described. Then, a sufficient condition for ensuring the disjoint property of the shortest paths is proposed. As shown in Section 3, this condition can be verified by solving a corresponding maximum bipartite matching problem. Section 4 shows the detailed computational results. In Section 5, this paper concludes with some remarks on the applications of our results. For simplicity, the set and multiset are used interchangeably in this paper.

2. Constructing disjoint shortest paths in an n-star

Suppose that s is the source node and $t_1, t_2, \ldots, t_m$ are m (not necessarily distinct) target nodes in an n-star, where $m \le n-1$, $n \ge 3$, and $s \notin \{t_1, t_2, \ldots, t_m\}$. Since star networks are node symmetric, we may assume $s = 12 \bullet\bullet\bullet [n]$, i.e. the origin, without loss of generality, where a number enclosed by a pair of square brackets, i.e. [ ], denotes a single digit. In this section, we first show the $O(mn)$ construction of m shortest paths $P_1, P_2, \ldots, P_m$ from s to $t_1, t_2, \ldots, t_m$, respectively, where $O(mn)$ is the optimal time complexity. Then, we propose a sufficient condition, which can be verified in $O(mn^{1.5})$ time, for ensuring the disjoint property of $P_1, P_2, \ldots, P_m$.

As described in Section 1, we may use $u = u_1 u_2 \bullet\bullet\bullet u_n$ to denote the node u of an n-star with label $u_1 u_2 \bullet\bullet\bullet u_n$, where $u_1 u_2 \bullet\bullet\bullet u_n$ is an n-digit permutation of $\{1, 2, \ldots, n\}$ such that $u_j \in \{1, 2, \ldots, n\}$ for all $1 \le j \le n$, and $u_j \ne u_k$ for all $1 \le j \ne k \le n$. For two nodes $u = u_1 u_2 \bullet\bullet\bullet u_n$ and $v = v_1 v_2 \bullet\bullet\bullet v_n$, they are connected by a link if $u_1 = v_k$ and $u_k = v_1$ for some $2 \le k \le n$, and $u_j = v_j$ for all $j \in \{2, 3, \ldots, n\} - \{k\}$. In other words, $u^{(k)} = v$ and $v^{(k)} = u$, where $u^{(k)}$ (resp. $v^{(k)}$) denotes the node obtained by swapping the first digit for the kth digit in u (resp. v). In order to describe the construction of $P_1, P_2, \ldots, P_m$, we introduce a terminology, named routing dimension, as defined below.

Definition 1. A *routing dimension* of a node u of an n-star is an integer r such that there exists a shortest path from s to u with $s^{(r)}$ as the following node of s, where $2 \le r \le n$.

2.1. Constructing all shortest $P_1, P_2, \ldots, P_m$

By convention, the n-digit label of node u was also expressed as a conjunction of disjoint non-trivial cycles $c_1^u c_2^u \cdots c_{|u|}^u$, and was written as $u = c_1^u c_2^u \cdots c_{|u|}^u$ for simplicity, where $|u|$ denotes the number of cycles in u. For all $1 \le k \le |u|$, we let $c_k^u = (c_{k,1}^u c_{k,2}^u \cdots c_{k,|c_k^u|}^u)$, where $u_{c_{k,|c_k^u|}^u} = c_{k,1}^u$ and $u_{c_{k,w}^u} = c_{k,w+1}^u$ for all $1 \le w \le |c_k^u| - 1$ and $|c_k^u| \ge 2$ denotes the length (i.e. the number of elements) of cycle c_k^u. For example, $u = 1342675 = (234)(567)$. It should be noted that u remains the same even if the order of its cycles is changed. In addition, the permutation represented by a cycle does not change after any cyclic shifts of the sequence of its elements.

Especially, when 1 is included in cycle c_k^u for some $1 \le k \le |u|$, we call it 1-cycle, and assume $k = 1$ and $c_{k,1}^u = c_{1,1}^u = 1$ without loss of generality. We have that if $u_1 = 1$, then there is no 1-cycle in u, otherwise $(u_1 \ne 1)$ there must be a 1-cycle in u with $c_{1,1}^u = 1$. By definition, it is easy to verify that $r \in \bigcup_{k=1}^{|u|} \{c_{k,w}^u | 1 \le w \le |c_k^u|\}$ if $u_1 = 1$, and $r \in \{c_{1,|c_1^u|}^u\} \cup \bigcup_{k=2}^{|u|} \{c_{k,w}^u | 1 \le w \le |c_k^u|\}$ otherwise $(u_1 \ne 1)$. For all $1 \le i \le m$, we use $t_{i,j}$ to denote the jth digit (from the left) of t_i, where $1 \le j \le n$. In addition, let $t_i = c_1^i c_2^i \cdots c_{|t_i|}^i$ and $c_k^i = (c_{k,1}^i c_{k,2}^i \cdots c_{k,|c_k^i|}^i)$ for all $1 \le k \le |t_i|$, and let $d_i = t_i$ if $t_{i,1} = 1$, and otherwise $(t_{i,1} \ne 1)$ $d_i = c_2^i c_3^i \cdots c_{|t_i|}^i$ and we may write $t_i = c_1^i d_i$ for simplicity. Let φ: $\{1, 2, \ldots, m\} \to \{2, 3, \ldots, n\}$ be a one-to-one mapping, where $m \le n-1$ and $n \ge 3$. Suppose