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Parameterized complexity of firefighting ☆

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ABSTRACT

The Firefighter problem is to place firefighters on the vertices of a graph to prevent a fire with known starting point from lighting up the entire graph. In each time step, a firefighter may be placed on an unburned vertex, permanently protecting it, and the fire spreads to all neighboring unprotected vertices of burning vertices. The goal is to let as few vertices burn as possible. In this paper, we consider a generalization of this problem, where at each time step $b \geqslant 1$ firefighters can be deployed. Our results answer several open questions raised by Cai et al. [8]. We show that this problem is W[1]-hard when parameterized by the number of saved vertices, protected vertices, and burned vertices. We also investigate several combined parameterizations for which the problem is fixed-parameter tractable. Some of our algorithms improve on previously known algorithms. We also establish lower bounds to polynomial kernelization.

1. Introduction

The Firefighter problem concerns a deterministic model of fire spreading through a graph via its edges. The problem has recently received considerable attention [13,18]. In the model, we are given a graph G with a vertex $s \in V(G)$. At time $t = 0$, the fire breaks out at s and vertex s starts burning. At each step $t \geqslant 1$, first the firefighter protects one vertex not yet on fire — this vertex remains permanently protected — and the fire then spreads from burning vertices to all unprotected neighbors of these vertices. The process stops when the fire cannot spread anymore. The goal is to find a strategy for the firefighter that minimizes the amount of burned vertices, or, equivalently, maximizes the number of saved, i.e. not burned, vertices.

It is known that the Firefighter problem is NP-hard, even when restricted to bipartite graphs [18] or trees of maximum degree three [14]. However, it is polynomial-time solvable on such trees if the root — the initially burning vertex — has degree two [18]. We refer to the survey [15] for an overview of further combinatorial results on the problem.

In this paper, we consider the parameterized complexity of the Firefighter problem on general graphs and trees where, at each time step, b firefighters can be deployed. Denote by b-Firefighter the generalization of the Firefighter problem

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with b firefighters. For any fixed budget $b \geqslant 2$, b-Firefighter was proved NP-hard for trees of maximum degree $b+3$, and polynomial-time solvable on trees of maximum degree $b+2$ provided that the fire breaks out at a vertex of degree at most $b+1$ [3].

The study of the problem from the perspective of parameterized complexity was initiated by Cai, Verbin, and Yang [8] for the case $b=1$. They considered the following three parameterized versions of the problem and obtained a number of parameterized algorithms on trees.

The first parameterization considered by Cai et al. in [8] is by the number of saved vertices.

> Saving k-Vertices **Parameter:** k
> **Input:** An undirected graph G, a vertex s, and two integers k and b.
> **Question:** Is there a strategy with respect to the budget b to save at least k vertices when a fire breaks out at s?

Cai et al. proved that Saving k-Vertices on trees has a polynomial kernel. They also gave a randomized algorithm solving Saving k-Vertices on trees in time $O(4^k+n)$, which can be derandomized to an $O(n+2^{O(k)})$-time algorithm.

The second parameterization considered by Cai et al. is by the number of burned vertices.

> Saving All But k-Vertices **Parameter:** k
> **Input:** An undirected graph G on n vertices, a vertex s, and two integers k and b.
> **Question:** Is there a strategy with respect to the budget b to save at least $n-k$ vertices when a fire breaks out at s?

For Saving All But k-Vertices on trees, Cai et al. gave a randomized algorithm of running time $O(4^k n)$, which can be derandomized to an $O(2^{O(k)} n \log n)$-time algorithm. They left as an open problem whether Saving All But k-Vertices has a polynomial kernel on trees.

The third parameterization investigated by Cai et al. is by the number of protected vertices, that is, the total number of vertices occupied by firefighters.

> Maximum k-Vertex Protection **Parameter:** k
> **Input:** An undirected graph G, a vertex s, and two integers k and b.
> **Question:** A strategy with respect to the budget b that saves the maximum number of vertices by protecting a total of at most k vertices when a fire breaks out at s?

For Maximum k-Vertex Protection on trees, Cai et al. gave a randomized algorithm of running time $O(k^{O(k)} n)$, which can be derandomized to an $O(k^{O(k)} n \log n)$-time algorithm. They left open whether the problem has a polynomial kernel on trees, and asked whether there is an algorithm solving the problem on trees in time $2^{o(k \log k)} n^{O(1)}$.

We will sometimes consider the decision variant of Maximum k-Vertex Protection.

> k-Vertex Protection **Parameter:** k
> **Input:** An undirected graph G, a vertex s, and three integers k, b and a.
> **Question:** Is there a strategy with respect to the budget b that saves at least a vertices by protecting a total of at most k vertices when a fire breaks out at s?

The unparameterized version of this problem is obviously NP-hard on trees of maximum degree three from the hardness of the Firefighter problem with $b=1$, and NP-hard on trees of maximum degree $b+3$ for any fixed $b \geqslant 2$ from the hardness of the b-Firefighter problem.

Our results. We resolve several open questions of Cai, Verbin, and Yang [8]. We also refine and extend some of the results of [8]. Fig. 1 summarizes our results.