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The spanning connectivity of the arrangement graphs

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HIGHLIGHTS

- We review some properties of the arrangement graph.
- We prove that there exists a set of disjoint paths in an incomplete arrangement graph.
- We prove that the arrangement graph $A_{n,k}$ is $k(n-k)^*$-connected for $n-k \geq 2$.

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ABSTRACT

A w-container $C_w(u, v)$ of a graph G between two distinct vertices u and v is a set of w disjoint paths between u and v. A w-container $C_w(u, v)$ is a w^*-container if every vertex of G is on some path in $C_w(u, v)$. A graph G is said to be w^*-connected if there exists a w^*-container between any two distinct vertices u and v. The connectivity of the arrangement graph $A_{n,k}$ is $k(n-k)$. In this paper, we prove that $A_{n,k}$ is $k(n-k)^*$-connected for $n-k \geq 2$.

1. Introduction

It is time of Big Data and Internet of Things that prevails in computer systems and information technology. In recent years, due to the popularization of mobile devices, the prevailing of social networks, and the improvement of cloud computing, enormous amount of data is produced in great speed. Internet of Things, for instance, every device is equipped with sensors. These devices are able to collect every kind of data extensively in large amount. Thus the parallel and distributed system is an important technique for developing Big Data. Related researches about interconnection network for the most parts have exactly applied to the parallel and distributed system. In a distributed computer system, a network structure represents the layout of the processors and the links. A graph usually represents the topological structure of a computer network, in which vertices represent processors and edges represent links between processors. The containers of the graphs do exist in information engineering design, telecommunication networks, and biological neural systems ([1,6] and its references). The research about the containers of graphs plays a key role in effective information transportation in terms of parallel routing design for large-scale networks.

For graph definitions and notations, we follow [8]. Let $G = (V, E)$ be a graph where V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. Two vertices u and v are adjacent if $(u, v) \in E$. We use $N(u)$ to denote the *neighbor* of a vertex u which is the set $\{v | (u, v) \in E\}$. We denote the *degree* of u by $deg(u) = |N(u)|$. We say that a graph G is *k-regular* if $deg(u) = k$ for every vertex u in G. A *path* $P = \langle v_1, v_2, \ldots, v_k \rangle$ between vertices v_1 and v_k is a sequence of adjacent vertices, where $v_1, v_2, \ldots, v_k$ are distinct vertices except that possibly $v_1 = v_k$. We also write the path P as $\langle v_1, v_2, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$, where Q is the path $v_i, v_{i+1}, \ldots, v_j$. Let $I(P) = V(P) - \{v_1, v_k\}$ be the set of the *internal vertices* of P. A set of paths $\{P_1, P_2, \ldots, P_k\}$ are *internally vertex-disjoint* (abbreviated as *disjoint*) if $I(P_i) \cap I(P_j) = \emptyset$ for any $i \neq j$. If a path contains all vertices of G, the path is a *hamiltonian path*. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct vertices of G. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. If a cycle traverses every vertex of G, it is a *hamiltonian cycle*. We say that a graph is *hamiltonian* if it contains a hamiltonian cycle. A graph G is *k-connected* if there exists a set of k internally disjoint paths between any two distinct vertices. Let u and v be two distinct vertices in a graph G. A *container* $C(u, v)$ between u and v in G is a set of disjoint paths joining u and v. A *w-container* $C_w(u, v)$ in G is a set of w disjoint paths between u

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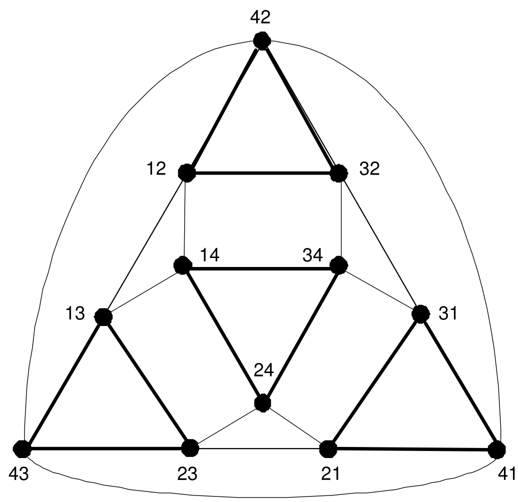

Fig. 1. The arrangement graph $A_{4,2}$.

and v. A w^*-*container* $C_{w^*}(u, v)$ in G is a w-container such that every vertex of G is on some path in $C_w(u, v)$. We say that G is w^*-connected if there exists a w^*-container between any two distinct vertices in G. Albert et al. proposed the 3^*-connected graphs in [2], which motivates the study of w^*-connected graph. The related works about the spanning connectivity have appeared in some literatures. In [11], Lin et al. proposed a sufficient condition for a graph to be w^*-connected. They proved that for all nonadjacent vertices u and v in a graph G, if $deg(u) + deg(v) \geq |V| + k$, G is w^*-connected for every $w \in \{1, 2, \dots, k + 2\}$. The spanning connectivity for the line graph and the power of a graph are proposed in [9,13], respectively. Some extensive properties of the spanning connectivity are discussed in [10,12].

The arrangement graph [4] was proposed by Day and Tripathi as a generalization of the star graph. It is more flexible in its size than the star graph. They also proved that the arrangement graph is hamiltonian [5]. Many research about the arrangement graph have been proposed. In [3], Cheng et al. enumerated the number of the shortest paths between any two vertices in the arrangement graph. The fault-tolerant properties about the arrangement graph are proposed in [14,17–19]. Let n and k be two positive integers with $n > k$. We use $\langle n\rangle$ and $\langle k\rangle$ to denote the sets $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, k\}$, respectively. The vertex set of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{p \mid p = p_1p_2\cdots p_k$ with $p_i \in \langle n\rangle$ for $1 \leq i \leq k$ and $p_i \neq p_j$ if $i \neq j\}$, and the edge set of $A_{n,k}$, $E(A_{n,k}) = \{(p, q) \mid p, q \in V(A_{n,k})$, p and q differ in exactly one position$\}$. Fig. 1 illustrates the arrangement graph $A_{4,2}$. By the definition of the arrangement graph $A_{n,k}$, $A_{n,k}$ is a $k(n-k)$-regular graph with $\frac{n!}{(n-k)!}$ vertices. Day and Tripathi proved that $A_{n,k}$ is vertex symmetric and edge symmetric in [4].

In this paper, we are interested in the spanning connectivity of the arrangement graphs. The connectivity of the arrangement graph $A_{n,k}$ is $k(n-k)$. We prove that $A_{n,k}$ is $k(n-k)^*$-connected for $n - k \geq 2$. In the following section, we give some properties about the arrangement graph $A_{n,k}$. In Section 3, we prove that there exists a $k(n-k)^*$-container between any two distinct vertices in $A_{n,k}$ for $n - k \geq 2$. Then we give our conclusion in the final section.

2. Some properties of the arrangement graphs

Suppose that $k > 1$. Let i and j be two integers with $1 \leq i, j \leq n$. We use $V(A^{(j:i)}_{n,k})$ to denote the set of all vertices with the jth position being i. That is, $V(A^{(j:i)}_{n,k}) = \{\mathbf{p} \mid \mathbf{p} = p_1p_2\dots p_k$ and $p_j = i\}$. For a fixed position j, $\{V(A^{(j:i)}_{n,k}) \mid 1 \leq i \leq n\}$ forms a partition of $V(A_{n,k})$. We use $A^{(j:i)}_{n,k}$ to denote the subgraph of $A_{n,k}$ induced by $V(A^{(j:i)}_{n,k})$. By the definition of the arrangement graphs, each $A^{(j:i)}_{n,k}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. Each $A^{(j:i)}_{n,k}$ represents a *subcomponent* of $A_{n,k}$, and we say that $A_{n,k}$ is decomposed into subcomponents according to the jth position. Note that each $A^{(j:i)}_{n,k}$ can be recursively decomposed into its smaller subcomponents. Suppose that $I \subseteq \{1, 2, \dots, n\}$. We use $A^{(j:I)}_{n,k}$ to denote the subgraph of $A_{n,k}$ induced by $\bigcup_{i\in I} V(A^{(j:i)}_{n,k})$. For simplicity, we use $A^i_{n,k}$ and $A^I_{n,k}$ to denote $A^{(k:i)}_{n,k}$ and $A^{(k:I)}_{n,k}$, respectively. That is, $V(A^i_{n,k}) = \{\mathbf{p} \mid \mathbf{p} = p_1p_2\dots p_k$ and $p_k = i\}$.

Let $E^{i,j}$ be the set of edges between $A^i_{n,k}$ and $A^j_{n,k}$. Suppose that $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$ is a faulty set of $A_{n,k}$. Suppose that $\mathbf{u}$ is a vertex in $A^i_{n,k}$ for some $i \in \langle n\rangle$ and $I \subseteq \{1, 2, 3, \dots, n\}$. Let $N(\mathbf{u})$ be the set of all neighbors of $\mathbf{u}$ in $A_{n,k}$. Moreover, we use $N^I(\mathbf{u})$ to denote the set of all neighbors of $\mathbf{u}$ in $A^I_{n,k}$. Particularly, we use $N^*(\mathbf{u})$ and $N^i(\mathbf{u})$ as an abbreviation of $N^{\langle n\rangle-\{i\}}(\mathbf{u})$ and $N^{\{i\}}(\mathbf{u})$, respectively. We call vertices in $N^*(\mathbf{u})$ the *outer neighbors* of $\mathbf{u}$. It follows from the definitions, $|N^i(\mathbf{u})| = (k - 1)(n - k)$ and $|N^*(\mathbf{u})| = n - k$. If there exists an outer neighbor of $\mathbf{u}$ in $A^j_{n,k}$, we say that $\mathbf{u}$ is adjacent to subcomponent $A^j_{n,k}$. Thus we define the *adjacent subcomponent* $AS(\mathbf{u})$ of $\mathbf{u}$ as $\{j \mid \mathbf{u}$ is adjacent to $A^j_{n,k}\}$. We need some basic properties of the arrangement graph. The following proposition follows directly from the definition of the arrangement graphs.

Proposition 1. *Let n and k be two positive integers with $n - k \geq 2$. Suppose that i and j are two distinct elements of $\langle n\rangle$, and H is one subcomponent of $A^i_{n,k}$ with the $(k - 1)$th position being h and the kth position being j for some $h \in \langle n\rangle - \{j\}$. Let $\{\mathbf{u}, \mathbf{u}', \mathbf{v}, \mathbf{v}'\}$ be four distinct vertices such that $\{\mathbf{u}, \mathbf{u}'\} \subset V(A^i_{n,k})$ and $\{\mathbf{v}, \mathbf{v}'\} \subset V(A^j_{n,k})$. Then $|E^{i,j}| = \frac{(n-2)!}{(n-k-1)!}$ and the number of edges between $A^i_{n,k}$ and H is $\frac{(n-3)!}{(n-k-1)!}$. Moreover, if $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ are distinct edges in $E^{i,j}$, then $\{\mathbf{u}, \mathbf{v}\} \cap \{\mathbf{u}', \mathbf{v}'\} = \emptyset$.*

Proposition 2. *Suppose that $n - k \geq 2$ and $\{i, j\} \subset \langle n\rangle$. Let $\mathbf{u} \in V(A^i_{n,k})$ and $\mathbf{v} \in V(A^j_{n,k})$ with $i \neq j$.*

(a) *If $k \geq 2$ and $d(\mathbf{u}, \mathbf{v}) = 1$, then there exists an index $p \in \langle k\rangle$ such that $\{\mathbf{u}, \mathbf{v}\} \subset V(A^{(p:q)}_{n,k})$ for some $q \in \langle n\rangle$.*
(b) *If $k \geq 3$ and $d(\mathbf{u}, \mathbf{v}) \leq 2$, then there exists an index $p \in \langle k\rangle$ such that $\{\mathbf{u}, \mathbf{v}\} \subset V(A^{(p:q)}_{n,k})$ for some $q \in \langle n\rangle$.*

Proof. Suppose that $k \geq 2$ and $d(\mathbf{u}, \mathbf{v}) = 1$. Let $\mathbf{u} = x_1x_2\cdots x_{k-1}i$ and $\mathbf{v} = x_1x_2\cdots x_{k-1}j$. Obviously, there exists an x_p such that $\{\mathbf{u}, \mathbf{v}\} \subset V(A^{(p:x_p)}_{n,k})$ for some $1 \leq p \leq k - 1$. Suppose that $k \geq 3$ and $d(\mathbf{u}, \mathbf{v}) = 2$. Without loss of the generality, we assume that there is a vertex $\mathbf{w} \in V(A^j_{n,k}) \cap N^j(\mathbf{v})$. Let $\mathbf{u} = x_1x_2\cdots x_{k-1}i$. Thus we have $\mathbf{w} = x_1x_2\cdots x_{k-1}j$. Since $\mathbf{w} \in N^j(\mathbf{v})$, $\mathbf{v} \in \{x_lx_2\cdots x_{k-1}j, x_1x_l\cdots x_{k-1}j, \dots, x_1x_2\cdots x_lj \mid x_l \neq j$ and $k \leq l \leq n\}$. Hence there exists an x_p such that $\{\mathbf{u}, \mathbf{v}\} \subset V(A^{(p:x_p)}_{n,k})$ for some $1 \leq p \leq k - 1$ if $k \geq 3$. □

Some studies on fault hamiltonicity and the hamiltonian connectivity of the arrangement graphs have been proposed in [4,7,15,16]. Some results are listed as follows.

Theorem 1 ([7]). *Suppose that $n-k \geq 2$ and $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$. Then $A_{n,k} - F$ is hamiltonian if $|F| \leq k(n - k) - 2$, and $A_{n,k} - F$ is hamiltonian connected if $|F| \leq k(n - k) - 3$.*

Theorem 2 ([7]). *Suppose that*

1. *$k \geq 3$, $n - k \geq 2$, and $I \subseteq \langle n\rangle$ with $|I| \geq 2$,*
2. *$F \subset V(A^I_{n,k})$ with $|F| \leq k(n - k) - 3$, and*
3. *$A^l_{n,k} - F$ is hamiltonian connected for each $l \in I$.*